On open-separating domination codes in graphs

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Outline

1. Identification problems in graphs
2. Properties of open-separating dominating codes
3. Open-separating dominating codes in several graph families
4. Benefit of reformulations as covering in hypergraphs
   - Analyzing the X-hypergraphs
   - From hypergraphs to clutters
   - About lower bounds on X-numbers
   - Pushing up lower bounds by polyhedral methods
5. Summary

Outline

Identification problems in graphs

Objective

Separate any two nodes of a graph by their unique neighborhoods in a suitably chosen (total-)dominating set (= **code**) of the graph.

Let $G = (V, E)$ be a graph and denote by $N(i)$ (resp. $N[i]$) the open (resp. closed) neighborhood of a node $i \in V$.

Domination properties

- $C \subseteq V$ is **dominating** if $N[i] \cap C$ are non-empty sets for all $i \in V$
- $C \subseteq V$ is **total-dominating** if $N(i) \cap C$ are non-empty sets for all $i \in V$

Separation properties

- $C \subseteq V$ is **closed-separating** if $N[i] \cap C$ are distinct sets for all $i \in V$
- $C \subseteq V$ is **open-separating** if $N(i) \cap C$ are distinct sets for all $i \in V$
- $C \subseteq V$ is **locating** if $N(i) \cap C$ are distinct sets for all $i \in V \setminus C$

Identification problems in graphs

Objective

Separate any two nodes of a graph by their unique neighborhoods in a suitably chosen (total-)dominating set (= **code**) of the graph.

The following identification problems have been studied in the literature by combining a domination and a separation property:

	domination	total-domination
closed-separation	identifying codes (**ID**-codes)	differentiating total-dominating codes (**DTD**-codes)
open-separation		open-locating dominating codes (**OLD**-codes)
location	locating-dominating codes (**LD**-codes)	locating total-dominating codes (**LTD**-codes)

Identification problems in graphs

Objective

Separate any two nodes of a graph by their unique neighborhoods in a suitably chosen (total-)dominating set (= **code**) of the graph.

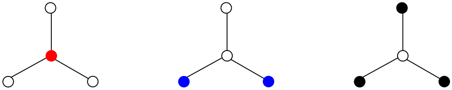
We here study the problem that arises by combining domination and open-separation:

	domination	total-domination
closed-separation	identifying codes (**ID**-codes)	differentiating total-dominating codes (**DTD**-codes)
open-separation	open-separating dominating codes (**OSD**-codes)	open-locating total-dominating codes (**OLD**-codes)
location	locating-dominating codes (**LD**-codes)	locating total-dominating codes (**LTD**-codes)

Some examples

For a graph $G = (V, E)$, a subset $C \subseteq V$ is

- dominating if $N[i] \cap C$ are non-empty sets for all $i \in V$
- open-separating if $N(i) \cap C$ are distinct sets for all $i \in V$
- an **open-separating dominating code** if it is dominating and open-separating



- total-dominating if $N(i) \cap C$ are non-empty sets for all $i \in V$
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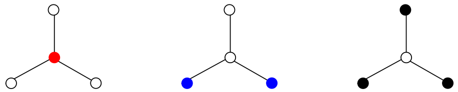

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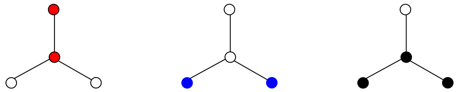

Identification problems in graphs

Let $\textsc{Codes} = \{ID, DTD, LD, LTD, OSD, OLD\}$.

X-code Problem

Given a graph G and $X \in \textsc{Codes}$, find an X-code of minimum cardinality $\gamma^X(G)$.

- There are manifold applications, including
  - placing sensors in sensor networks
  - detecting faulty processors in multi-processor networks
- The related decision problems are all NP-complete.
- Typical lines of attack: for special graphs G,
  - give a closed formula or (lower or upper) bounds for $\gamma^X(G)$
  - design (combinatorial) algorithms to determine $\gamma^X(G)$
- Active research area, see the bibliography maintained by Jean and Lobstein: `https://dragazo.github.io/bibdom/main.pdf`

Outline

Existence

Relations to other X-codes

Reformulation as covering in hypergraphs

To study X-problems under a **unifying point of view**, reformulate the X-problems in a graph as covering problem in suitably constructed hypergraphs!

Covers of hypergraphs

Let $\mathcal{H} = (V, \mathcal{F})$ be a hypergraph.

- A **cover** of $\mathcal{H}$ is a subset $C \subseteq V$ satisfying $C \cap F \neq \emptyset$ for all $F \in \mathcal{F}$.
- The **covering number** $\tau(\mathcal{H})$ is the minimum cardinality of a cover C of $\mathcal{H}$.

X-hypergraph $\mathcal{H}_{\mathrm{X}}(G)$

For a graph $G = (V, E)$ and $X \in \textsc{Codes}$, the **X-hypergraph** $\mathcal{H}_{\mathrm{X}}(G) = (V, \mathcal{F}_{\mathrm{X}})$ satisfies that $C \subseteq V$ is an X-code of G **if and only if** C is a cover of $\mathcal{H}_{\mathrm{X}}(G)$.

We have by construction:

Observation on X-numbers

For any graph G and $X \in \textsc{Codes}$, we have $\gamma^{\mathrm{X}}(G) = \tau(\mathcal{H}_{\mathrm{X}}(G))$.

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Reformulation as covering in hypergraphs

- $\mathcal{H}_{\mathrm{X}}(G)$ clearly needs to contain the neighborhoods of all nodes of G.
- To encode that the intersections of an X-code C with the neighborhoods are **unique**, use the symmetric differences of the neighborhoods!

Theorem

A code C of a graph G is

- **closed-separating** if and only if $(N[i] \Delta N[j]) \cap C \neq \emptyset$ for all pairs of distinct nodes i, j of G,
- **open-separating** if and only if $(N(i) \Delta N(j)) \cap C \neq \emptyset$ for all pairs of distinct nodes i, j of G,
- **locating** if and only if
  - $(N(i) \Delta N(j)) \cap C \neq \emptyset$ for all pairs of **adjacent** nodes i, j of G and
  - $(N[i] \Delta N[j]) \cap C \neq \emptyset$ for all pairs of **non-adjacent** nodes i, j of G.
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Reformulation as covering in hypergraphs

Corollary

The X-hypergraphs $\mathcal{H}_X(G) = (V, \mathcal{F}_X)$ of the X-problems for $X \in \textsc{Codes}$ contain the following combinations of sets of hyperedges:

ID	DTD	LD	LTD	PD	PTD	OSD	OLD
$N[G]$	$N(G)$	$N[G]$	$N(G)$	$N[G]$	$N(G)$	$N[G]$	$N(G)$
$\triangle_a[G]$	$\triangle_a[G]$	$\triangle_a(G)$	$\triangle_a(G)$	$\triangle_a[G]$	$\triangle_a[G]$	$\triangle_a(G)$	$\triangle_a(G)$
$\triangle_n[G]$	$\triangle_n[G]$	$\triangle_n[G]$	$\triangle_n[G]$	$\triangle_n(G)$	$\triangle_n(G)$	$\triangle_n(G)$	$\triangle_n(G)$

- $N[G]$ (resp. $N(G)$) denotes the set of all closed (resp. open) **neighborhoods** of nodes in G,
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Reformulation as covering in hypergraphs

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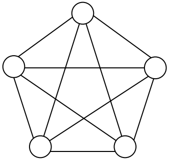
Hardness

Outline

Examples of X-hypergraphs: Cliques

A **clique** is a graph $K_n = (V, E)$ with

- n nodes in $V = \{1, \dots, n\}$,
- all possible edges.



LD/OSD-hypergraph	LTD/OLD-hypergraph
$N[i] = V$	$V \setminus \{i\} = N(i)$
$N(i) \Delta N(j) = \{i, j\}$	$\{i, j\} = N(i) \Delta N(j)$
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LD/OSD-codes of K_n	LTD/OLD-codes of K_n
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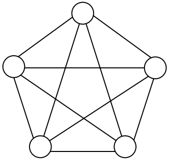

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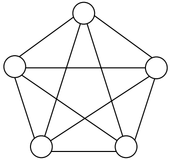

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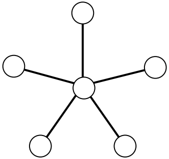
Examples of X-hypergraphs: Stars



A **star** is a graph $K_{1,n} = (V, E)$ with

- $n + 1$ nodes in $V = \{0, 1, \dots, n\}$,
- edges $0i$ for $1 \leq i \leq n$.

ID-hypergraph	LTD-hypergraph
$N[0] = V$ $N[i] = \{0, i\}$	$\{1, \dots, n\} = N(0)$ $\{0\} = N(i)$
$N[0] \Delta N[i] = \{1, \dots, n\} \setminus \{i\}$	$V = N(0) \Delta N(i)$
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ID-codes of $K_{1,n}$	LTD-codes of $K_{1,n}$
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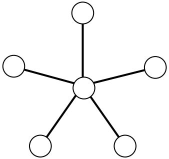

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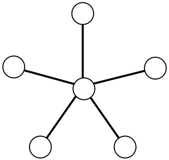

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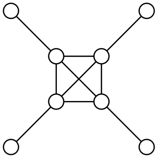
Examples of X-hypergraphs: Thin spiders

A **thin spider** is a graph $H_k = (S \cup Q, E)$ where

- $|S| = |Q| = k$ with $k \geq 3$,
- S induces a stable set, Q induces a clique,
- s_i is adjacent to q_j if and only if $i = j$.



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LD-codes of H_k	OSD-codes of H_k
S, Q are LD-codes	Q is an OSD-code

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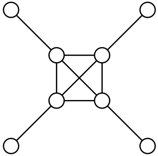

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LD-codes of H_k	OSD-codes of H_k
S, Q are LD-codes	Q is an OSD-code

Examples of X-hypergraphs: Thin spiders

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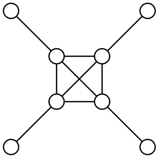

LD-hypergraph	OSD-hypergraph
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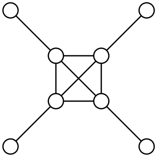

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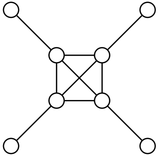

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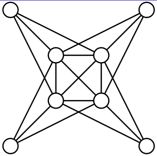

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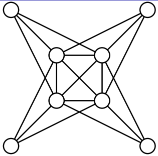

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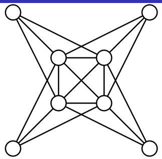

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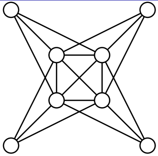

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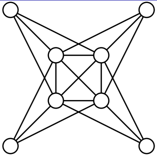

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Outline

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About the existence of X-codes in graphs

Observation

A hypergraph $\mathcal{H} = (V, \mathcal{F})$ has **no cover** if and only if $\emptyset \in \mathcal{F}$.

Corollary

There is no X-code in a graph G with $\mathcal{H}_{\mathrm{X}}(G)$ involving

- $N(G)$ if G has isolated nodes i with $N(i) = \emptyset$;
- $\triangle_a[G]$ if G has closed twins: adjacent nodes i, j with $N[i] = N[j]$ (and thus $N[i] \Delta N[j] = \emptyset$);
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ID	DTD	LD	LTD	PD	PTD	OSD	OLD
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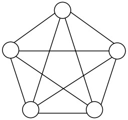
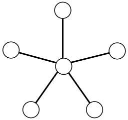
Example: Closed and open twins



Cliques K_n have closed twins and no X-codes for $X \in \{ID, DTD, PD, PTD\}$

ID-hypergraph:

$N[G]$	$N[i]$	$=$	V
$\triangle_a[G]$	$N[i]\Delta N[j]$	$=$	$\emptyset$
$\triangle_n[G]$		$-$	



Stars $K_{1,n}$ have open twins and no X-codes for $X \in \{PD, PTD, OSD, OLD\}$

PD-hypergraph:

$N[G]$	$N[0]$	$=$	V
	$N[i]$	$=$	$\{0, i\}$
$\triangle_a[G]$	$N[0]\Delta N[i]$	$=$	$V \setminus \{0, i\}$
$\triangle_n(G)$	$N(i)\Delta N(j)$	$=$	$\emptyset$

About the relations of X-numbers

Consider hypergraphs $\mathcal{H} = (V, \mathcal{F})$ and $\mathcal{H}' = (V, \mathcal{F}')$ on the same node set V. We say $\mathcal{H} \prec \mathcal{H}'$ if, for each F' of $\mathcal{H}'$, there exists F of $\mathcal{H}$ such that $F \subseteq F'$.

Lemma

For $\mathcal{H} = (V, \mathcal{F})$ and $\mathcal{H}' = (V, \mathcal{F}')$ with $\mathcal{H} \prec \mathcal{H}'$, we have $\tau(\mathcal{H}') \leq \tau(\mathcal{H})$.

Proof: Consider a cover $C \subseteq V$ of $\mathcal{H}$.

- For every F' of $\mathcal{H}'$, there is F of $\mathcal{H}$ such that $F \subseteq F'$ by $\mathcal{H} \prec \mathcal{H}'$.
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- Hence, C is also a cover of $\mathcal{H}'$ and the assertion follows. □

N.B. For all graphs, it is clear that

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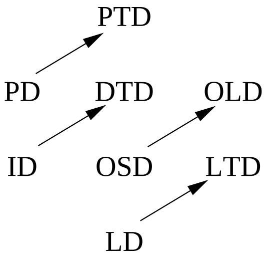
If for a graph $G = (V, E)$ *and two problems* $X, X' \in$ Codes *the hypergraphs* $\mathcal{H}_{\mathrm{X}}(G) = (V, \mathcal{F}_{\mathrm{X}})$ *and* $\mathcal{H}'_{\mathrm{X}}(G) = (V, \mathcal{F}'_{\mathrm{X}})$ *only differ in the*

- *neighborhoods such that* $N(G) \subset \mathcal{F}_{\mathrm{X}}$ *and* $N[G] \subset \mathcal{F}_{\mathrm{X}'}$,
- *symmetric differences with* $\triangle_a[G] \subset \mathcal{F}_{\mathrm{X}}$ *and* $\triangle_a(G) \subset \mathcal{F}_{\mathrm{X}'}$,
- *symmetric differences with* $\triangle_n(G) \subset \mathcal{F}_{\mathrm{X}}$ *and* $\triangle_n[G] \subset \mathcal{F}_{\mathrm{X}'}$,

then $\mathcal{H}_{\mathrm{X}}(G) \prec \mathcal{H}_{\mathrm{X}'}(G)$ *follows in all cases and implies* $\gamma^{X'}(G) \leq \gamma^{\mathrm{X}}(G)$.



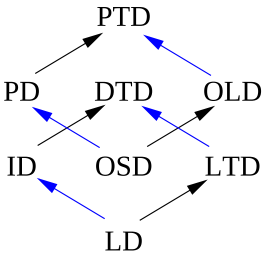
About the relations of X-numbers

Corollary

If for a graph $G = (V, E)$ and two problems $X, X' \in$ $\textsc{Codes}$ the hypergraphs $\mathcal{H}_{\mathrm{X}}(G) = (V, \mathcal{F}_{\mathrm{X}})$ and $\mathcal{H}'_{\mathrm{X}}(G) = (V, \mathcal{F}'_{\mathrm{X}})$ only differ in the

- *neighborhoods such that $N(G) \subset \mathcal{F}_{\mathrm{X}}$ and $N[G] \subset \mathcal{F}_{\mathrm{X}'}$,*
- *symmetric differences with $\triangle_a[G] \subset \mathcal{F}_{\mathrm{X}}$ and $\triangle_a(G) \subset \mathcal{F}_{\mathrm{X}'}$,*
- *symmetric differences with $\triangle_n(G) \subset \mathcal{F}_{\mathrm{X}}$ and $\triangle_n[G] \subset \mathcal{F}_{\mathrm{X}'}$,*

then $\mathcal{H}_{\mathrm{X}}(G) \prec \mathcal{H}_{\mathrm{X}'}(G)$ follows in all cases and implies $\gamma^{X'}(G) \leq \gamma^{\mathrm{X}}(G)$.



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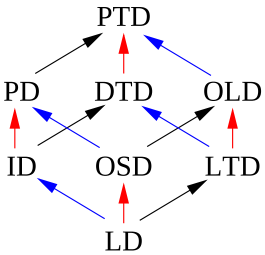

Outline

From X-hypergraphs to X-clutters

To calculate the X-numbers of the studied graphs G, we construct the X-hypergraphs $\mathcal{H}_X(G)$ and determine their covering numbers $\tau(\mathcal{H}_{\mathrm{X}}(G))$.

Redundant hyperedges

Consider a hypergraph $\mathcal{H} = (V, \mathcal{F})$.

- If there are two hyperedges $F, F' \in \mathcal{F}$ with $F \subseteq F'$, then $F \cap C \neq \emptyset$ also implies $F' \cap C \neq \emptyset$ for every $C \subseteq V$.
- F' is **redundant** as $(V, \mathcal{F} - \{F'\})$ suffices to encode the covers of $\mathcal{H}$.

Hence, only non-redundant hyperedges of $\mathcal{H}_{\mathrm{X}}(G)$ need to be considered in order to determine $\tau(\mathcal{H}_{\mathrm{X}}(G))$ and thus $\gamma^{\mathrm{X}}(G)$.

X-clutters $\mathcal{C}_{\mathrm{X}}(G)$

The **X-clutter** $\mathcal{C}_{\mathrm{X}}(G)$ of the graph G is obtained from $\mathcal{H}_{\mathrm{X}}(G)$ by removing all redundant hyperedges. For any graph $G = (V, E)$ and $X \in \textsc{Codes}$, we have

$$\gamma^{\mathrm{X}}(G) = \tau(\mathcal{C}_{\mathrm{X}}(G)) = \tau(\mathcal{H}_{\mathrm{X}}(G))$$

From X-hypergraphs to X-clutters

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About X-clutters

The X-clutter $\mathcal{C}_{\mathrm{X}}(G)$ of a graph G may turn out to be a (hyper)graph already studied in the literature concerning its covering number.

Stable sets and covers in graphs

- A subset of non-adjacent nodes of a graph is a **stable set**.
- The size of a maximum stable set of G is denoted by $\alpha(G)$.
- For any graph G with n nodes, we have $\tau(G) = n - \alpha(G)$.

Example: For a complete graph K_n, we have $\alpha(K_n) = 1$ and $\tau(K_n) = n - 1$.

Complete q-roses of order n

- The hypergraph $\mathcal{R}_n^q = (V, \mathcal{E})$ is a **complete** q**-rose of order** n, when
  - $V = \{1, \ldots, n\}$ and
  - $\mathcal{E}$ contains all q-element subsets of V for $2 \leq q < n$.
- For a complete q-rose of order n, we have $\tau(\mathcal{R}_n^q) = n - q + 1$.

Example: For $q = 2$, $\mathcal{R}_n^q$ is the complete graph K_n and $\tau(K_n) = n - 2 + 1 = n - 1$.

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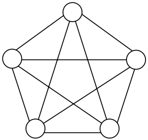
Examples of X-clutters: Cliques



A **clique** is a graph $K_n = (V, E)$ with

- n nodes in $V = \{1, \dots, n\}$,
- all possible edges.

LD/OSD-hypergraph	LTD/OLD-hypergraph
$N[i] = V$	$V \setminus \{i\} = N(0)$
$N(i) \Delta N(j) = \{i, j\}$	$\{i, j\} = N(i) \Delta N(j)$
$-$	$-$

LD/OSD-codes of K_n	LTD/OLD-codes of K_n
$V \setminus \{i\}$ for $1 \leq i \leq n$	$V \setminus \{i\}$ for $1 \leq i \leq n$
$V = \{1, \dots, n\}$	$V = \{1, \dots, n\}$

- In all cases, the neighborhoods are redundant for $n \geq 3$.
- $\mathcal{C}_{\mathrm{X}}(K_n) = \mathcal{R}_n^2 = K_n$ for all $X \in \{LD, LTD, OSD, OLD\}$.
- $\gamma^{\mathrm{X}}(K_n) = \tau(\mathcal{C}_{\mathrm{X}}(K_n)) = \tau(\mathcal{R}_n^2) = \tau(K_n) = n - 1$.

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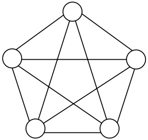

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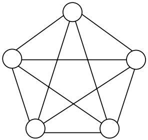

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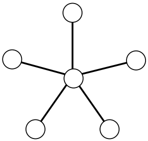
Examples of X-clutters: Stars

A **star** is a graph $K_{1,n} = (V, E)$ with



- $n + 1$ nodes in $V = \{0, 1, \ldots, n\}$,
- edges $0i$ for $1 \leq i \leq n$.

	ID-hypergraph	LTD-hypergraph	
$N[0] =$	V	$\{1, \ldots, n\}$	$= N(0)$
$N[i] =$	$\{0, i\}$	$\{0\}$	$= N(i)$
$N[0]\Delta N[i] =$	$\{1, \ldots, n\} \setminus \{i\}$	V	$= N(0)\Delta N(i)$
$N[j]\Delta N[i] =$	$\{i, j\}$	$\{i, j\}$	$= N[j]\Delta N[i]$

	ID-clutter of $K_{1,n}$	LTD-clutter of $K_{1,n}$	
$\mathcal{C}_{ID}(K_{1,n}) =$	$\mathcal{R}^2_{1+n} = K_{1+n}$	$\mathcal{C}_{LTD}(K_{1,n})$	$= K_1 \cup K_n$
$\gamma^{\mathrm{ID}}(K_{1,n}) =$	$\tau(K_{1+n}) = n$	$\gamma^{\mathrm{LTD}}(K_{1,n})$	$= 1 + \tau(K_n) = n$

A **star** is a graph $K_{1,n} = (V, E)$ with

- $n + 1$ nodes in $V = \{0, 1, \ldots, n\}$,
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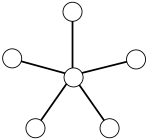

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$N[i] =$	$\{0, i\}$	$\{0\}$	$= N(i)$
$N[0]\Delta N[i] =$	$\{1, \ldots, n\} \setminus \{i\}$	V	$= N(0)\Delta N(i)$
$N[j]\Delta N[i] =$	$\{i, j\}$	$\{i, j\}$	$= N[j]\Delta N[i]$

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$\mathcal{C}_{ID}(K_{1,n}) =$	$\mathcal{R}^2_{1+n} = K_{1+n}$	$\mathcal{C}_{LTD}(K_{1,n})$	$= K_1 \cup K_n$
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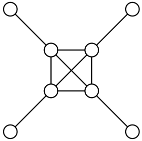
Examples of X-clutters: Thin spiders

A **thin spider** is a graph $H_k = (S \cup Q, E)$ where

- $|S| = |Q| = k$ with $k \geq 4$,
- S induces a stable set, Q induces a clique,
- s_i is adjacent to q_j if and only if $i = j$.

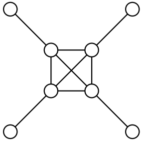


LD-hypergraph	OSD-hypergraph
$N[s_i] = \{s_i, q_i\}$	$\{s_i, q_i\} = N[s_i]$
$N[q_i] = Q \cup \{s_i\}$	$Q \cup \{s_i\} = N[q_i]$
$N(s_i)\Delta N(q_i) = Q \cup \{s_i\}$	$Q \cup \{s_i\} = N(s_i)\Delta N(q_i)$
$N(q_i)\Delta N(q_j) = \{q_i, q_j\} \cup \{s_i, s_j\}$	$\{q_i, q_j\} \cup \{s_i, s_j\} = N(q_i)\Delta N(q_j)$
$N[s_i]\Delta N[s_j] = \{q_i, q_j\} \cup \{s_i, s_j\}$	$\{q_i, q_j\} = N(s_i)\Delta N(s_j)$
$N[s_i]\Delta N[q_j] = \{s_i, s_j\} \cup (Q \setminus \{q_i\})$	$\{s_i\} \cup (Q \setminus \{q_i, q_j\}) = N(s_i)\Delta N(q_j)$

LD-clutter of H_k	OSD-clutter of H_k
$\mathcal{C}_{LD}(H_k) = k \cdot K_2$	$\mathcal{C}_{OSD}(H_k) = H_k$
$\gamma^{\mathrm{LD}}(H_k) = k \cdot 1 = k$	$\gamma^{\mathrm{LTD}}(H_k) = 2k - \alpha(H_k) = k$

Examples of X-clutters: Thin spiders

A **thin spider** is a graph $H_k = (S \cup Q, E)$ where



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LD-hypergraph		OSD-hypergraph	
$N[s_i]$	$= \{s_i, q_i\}$	$\{s_i, q_i\}$	$= N[s_i]$
$N[q_i]$	$= Q \cup \{s_i\}$	$Q \cup \{s_i\}$	$= N[q_i]$
$N(s_i)\Delta N(q_i)$	$= Q \cup \{s_i\}$	$Q \cup \{s_i\}$	$= N(s_i)\Delta N(q_i)$
$N(q_i)\Delta N(q_j)$	$= \{q_i, q_j\} \cup \{s_i, s_j\}$	$\{q_i, q_j\} \cup \{s_i, s_j\}$	$= N(q_i)\Delta N(q_j)$
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LD-clutter of H_k		OSD-clutter of H_k	
$\mathcal{C}_{LD}(H_k)$	$= k \cdot K_2$	$\mathcal{C}_{OSD}(H_k)$	$= H_k$
$\gamma^{\mathrm{LD}}(H_k)$	$= k \cdot 1 = k$	$\gamma^{\mathrm{LTD}}(H_k)$	$= 2k - \alpha(H_k) = k$

Examples of X-clutters: Thick spiders

A **thick spider** is a graph $\overline{H}_k = (S \cup Q, E)$ where

- $|S| = |Q| = k$ with $k \geq 4$,
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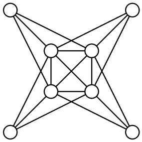

LD-hypergraph	DTD-hypergraph
$N[s_i] = \{s_i\} \cup (Q \setminus \{q_i\})$	$Q \setminus \{q_i\} = N(s_i)$
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$N(s_i) \Delta N(q_j) = \{q_i, q_j\} \cup (S \setminus \{s_j\})$	$\{q_i\} \cup (S \setminus \{s_i, s_j\}) = N[s_i] \Delta N[q_j]$
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LD-clutter of $\overline{H}_k$	DTD-clutter of $\overline{H}_k$
$\mathcal{C}_{LD}(\overline{H}_k) =$ rather involved!	$\mathcal{C}_{DTD}(\overline{H}_k) = \mathcal{R}_k^{k-1} \cup \mathcal{R}_k^2$
$\gamma^{\mathrm{LD}}(\overline{H}_k) = ???$	$\gamma^{\mathrm{DTD}}(\overline{H}_k) = 2+(k-1)=k+1$

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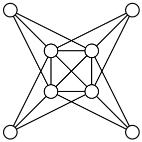

LD-hypergraph		DTD-hypergraph	
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Outline

Lower bounds from (total-)domination

Observation

For each X-problem involving

- **domination**, the dominating number $\gamma(G)$
- **total-domination**, the total-dominating number $\gamma^{t}(G)$

is a lower bound on $\gamma^{X}(G)$.

How good can these bounds be?

Clique K_n

$\gamma(K_n)$	$=$	1
$\gamma^{t}(K_n)$	$=$	2
$\gamma^{LD}(K_n)$	$=$	n-1

Star $K_{1,n}$

$\gamma(K_{1,n})$	$=$	1
$\gamma^{t}(K_{1,n})$	$=$	2
$\gamma^{LTD}(K_{n1,})$	$=$	n-1

Lower bounds from (total-)domination

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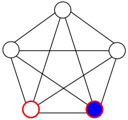

$\gamma(K_n)$	$=$	1
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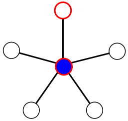
$\gamma^{\mathrm{LD}}(K_n)$	$=$	n-1



Star $K_{1,n}$

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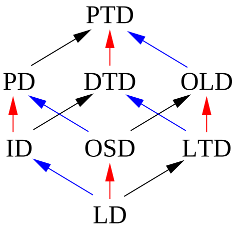
Lower bounds from other X-numbers



How large can the gaps be?

Theorem

- $\gamma^{\mathrm{OLD}}(G) - 1 \leq \gamma^{\mathrm{OSD}}(G) \leq \gamma^{\mathrm{OLD}}(G)$
- $\gamma^{\mathrm{PTD}}(G) - 1 \leq \gamma^{\mathrm{PD}}(G) \leq \gamma^{\mathrm{PTD}}(G)$

The other gaps between related X-numbers can be arbitrarily large, e.g.

- $\gamma^{\mathrm{LTD}}(H_k) = k < 2k - 1 = \gamma^{\mathrm{DTD}}(H_k)$
- $\gamma^{\mathrm{ID}}(\overline{H}_k) = k < 2k - 2 = \gamma^{\mathrm{PD}}(\overline{H}_k)$

Lower bounds from the dual problems

For the studied X-problems,

- there is no straightforward "direct" dual problem,
- but from the covering in the X-hypergraphs!

Matchings in hypergraphs

Consider a hypergraph $\mathcal{H} = (V, \mathcal{E})$.

- A subset $\mathcal{F} \subseteq \mathcal{E}$ is a **matching** of $\mathcal{H}$ if $F \cap F' = \emptyset$ for all $F, F' \in \mathcal{F}$.
- The **matching number** $\nu(\mathcal{H})$ is the cardinality of a maximum matching.

Matchings and covers in hypergraphs are dual objects:

Matchings and covers

- For every hypergraph $\mathcal{H}$, we have $\nu(\mathcal{H}) \leq \tau(\mathcal{H})$.
- A hypergraph $\mathcal{H}$ **packs** if $\nu(\mathcal{H}) = \tau(\mathcal{H})$ holds.

Lower bounds from the dual problems

For the studied X-problems,

- there is no straightforward "direct" dual problem,
- but from the covering in the X-hypergraphs!

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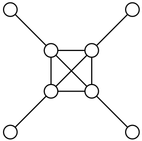
Example: X-clutters that pack

A **thin spider** is a graph $H_k = (S \cup Q, E)$ where

- $|S| = |Q| = k$ with $k \geq 4$,
- S induces a stable set, Q induces a clique,
- s_i is adjacent to q_j if and only if $i = j$.



LD-hypergraph		OSD-hypergraph	
$N[s_i] =$	$\{s_i, q_i\}$	$\{s_i, q_i\}$	$= N[s_i]$
$N[q_i] =$	$Q \cup \{s_i\}$	$Q \cup \{s_i\}$	$= N[q_i]$
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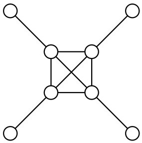

	LD-clutter	OSD-clutter
	$N[s_i] \; = \; \{s_i, q_i\}$	$\{s_i, q_i\} \; = \; N[s_i]$
	–	–
	–	$\{q_i, q_j\} \; = \; N(s_i) \Delta N(s_j)$

LD-clutter of H_k is $\mathcal{C}_{LD}(H_k) = k \cdot K_2$	OSD-clutter of H_k is $\mathcal{C}_{OSD}(H_k) = H_k$
$N[s_i] = \{s_i, q_i\}$ induce a matching: $\nu(\mathcal{C}_{LD}(H_k)) = k$	$N[s_i] = \{s_i, q_i\}$ induce a matching: $\nu(\mathcal{C}_{OSD}(H_k)) \geq k$
S and Q are LD-codes: $\tau(\mathcal{C}_{LD}(H_k)) \leq k$	Q is an OSD-code: $\tau(\mathcal{C}_{OSD}(H_k)) \leq k$

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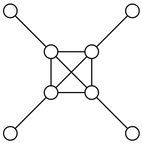

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Outline

Covering in X-clutters as integer program

Consider an X-clutter $\mathcal{C}_X(G) = (V, \mathcal{F})$. We know that

$$\tau(\mathcal{C}_X(G)) = \min\{|C| : C \subseteq V, C \cap F \neq \emptyset \ \forall F \in \mathcal{F}\}$$

Integer program to determine $\tau(\mathcal{C}_X(G))$

Using variables $x_i \in \{0,1\}$ for all $i \in V$ to encode covers $C \subseteq V$, we get

$$\begin{array}{rcccl} \tau(\mathcal{C}_X(G)) = \min \mathbf{x}(V) & = & \sum_{i \in V} x_i & & \\ s.t.\ \mathbf{x}(F) & = & \sum_{i \in F} x_i & \geq & 1 \ \ \forall F \in \mathcal{F} \\ & & x_i & \in & \{0,1\} \end{array}$$

Example. The LD-clutter of thin spiders $H_k = (S \cup Q, E)$ is composed of $\{s_i, q_i\}$ for $1 \leq i \leq k$.

$$\begin{array}{rcll} \tau(\mathcal{C}_{LD}(G)) = \min \sum_{i \in V} x_i & & & \\ s.t.\ x_{s_i} + x_{q_i} & \geq & 1 & \forall i \in \{1, \ldots, k\} \\ x_{s_i},\ \ x_{q_i} & \in & \{0,1\} & \forall i \in \{1, \ldots, k\} \end{array}$$

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About covering polyhedra

For any hypergraph $\mathcal{H} = (V, \mathcal{E})$, we have

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$$P(\mathcal{H}) = \text{conv}\{\mathbf{x} \in \{0, 1\}^V : \sum_{i \in F} x_i \geq 1 \ \ \forall F \in \mathcal{E}\}.$$

We are particularly interested in finding the **full rank constraint**

$$\mathbf{x}(V) = \sum_{i \in V} x_i \geq \tau(\mathcal{H})$$

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Enhancing integer programs

Consider an X-clutter $\mathcal{C}_X(G) = (V, \mathcal{F})$ and a maximum matching $\mathcal{F}' \subseteq \mathcal{F}$. Adding up the constraints

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for all $F \in \mathcal{F}'$ yields $\mathbf{x}(V) = \sum_{i \in V} x_i \geq \nu(\mathcal{C}_X(G))$.

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 & & & x_{s_4} & & & & +x_{q_4} & \geq & 1 \\
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x_{s_1} & +x_{s_2} & +x_{s_3} & +x_{s_4} & +x_{q_1} & +x_{q_2} & +x_{q_3} & +x_{q_4} & \geq & 4
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Chvátal-Gomory cuts for covering formulations

If $\mathbf{a}^T\mathbf{x} \geq b$ is a linear combination of constraints from a covering formulation with $b \notin \mathbf{Z}$, then every cover also satisfies $\mathbf{a}^T\mathbf{x} \geq \lceil b \rceil$, called **Chvátal-Gomory cut**.

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Examples of Chvátal-Gomory cuts

The LD-clutter of cliques K_n is $\mathcal{C}_{LD}(K_n) = \mathcal{R}_n^2 = K_n$. For $n = 3$, we get

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Taking a linear combination and scaling by $\frac{1}{2}$ yields a Chvátal-Gomory cut:

$$\begin{array}{ccccccc} 2x_1 & + & 2x_2 & + & 2x_3 & \geq & 3 \\ \hline x_1 & + & x_2 & + & x_3 & \geq & \lceil \frac{3}{2} \rceil = 2 \end{array}$$

Theorem

For a complete q-rose $\mathcal{R}_n^q = (V, \mathcal{E})$,

$$\mathbf{x}(V') = \sum_{i \in V'} x_i \geq |V'| - 1$$

is a Chvátal-Gomory cut for all $V' \subseteq V$ with $|V'| > q$.
In particular, $\mathbf{x}(V) \geq n - q + 1$ holds and implies $\tau(\mathcal{R}_n^q) = n - q + 1$.

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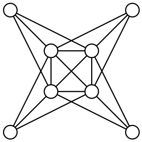

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LD-clutter of $\overline{H}_k$		DTD-clutter of $\overline{H}_k$	
$\mathcal{C}_{LD}(\overline{H}_k)$	$=$ rather involved!	$\mathcal{C}_{DTD}(\overline{H}_k) =$	$\mathcal{R}_k^{k-1} \cup \mathcal{R}_k^2$
$\gamma^{\mathrm{LD}}(\overline{H}_k)$	$=$???	$\gamma^{\mathrm{DTD}}(\overline{H}_k) =$	$2+(k-1)=k+1$

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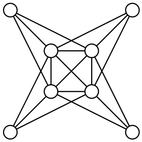

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We can use $\{q_i, q_j\} \cup \{s_i, s_j\}$ to build Chvátal-Gomory cuts:

Theorem

For the LD-problem in thick spiders $\overline{H}_k = (S \cup Q, E)$, we obtain

$$\sum_{i \in I} x_{s_i} + \sum_{i \in I} x_{q_i} \geq |I| - 1$$

as Chvátal-Gomory cut for all $I \subseteq \{1,\ldots,k\}$ with $|I| > 3$.

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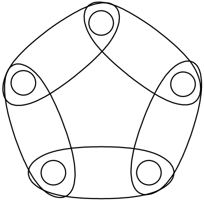
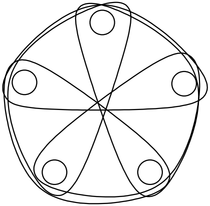
More interesting substructures of X-clutters

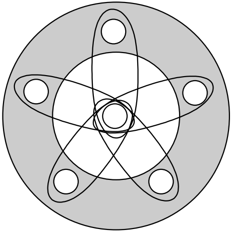
Besides complete q-roses $\mathcal{R}_n^q$, many other interesting substructures of hypergraphs have been studied in the reach literature about covering problems and polyhedra:



cycles



circulants



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All of them can be used

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even if the whole hypergraph is not well-structured!

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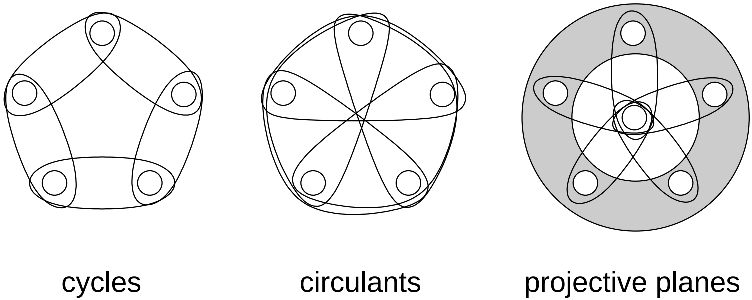

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Outline

Concluding Remarks

The studied X-problems are all NP-hard, typical lines of attack: for special graphs,

- give a closed formula or bounds for $\gamma^X(G)$;
- design algorithms to determine $\gamma^X(G)$ in polynomial time.

Study X-problems from a unifying point of view

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Benefit:

- mixing all (= combinatorial and polyhedral) techniques may enable us to solve large-scale instances issued from real applications!

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