A linear algorithm for radio k-coloring of powers of paths having small diameter

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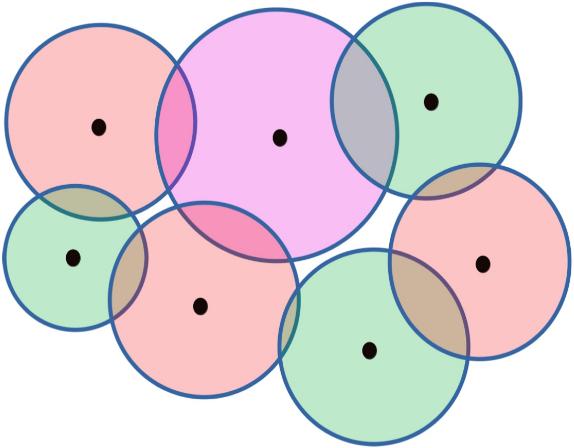
Channel Assignment Problem



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ℓ**-radio** k**-coloring** c of a graph G is $c : V(G) \to \{0, 1, 2, \ldots, \ell\}$ such that:

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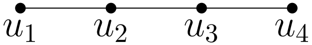

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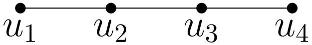

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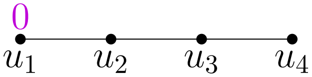

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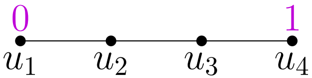

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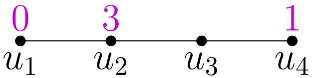

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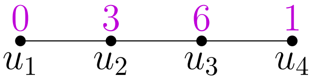

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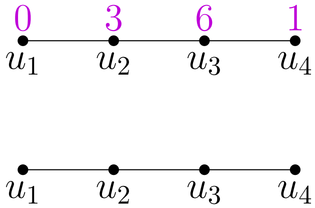

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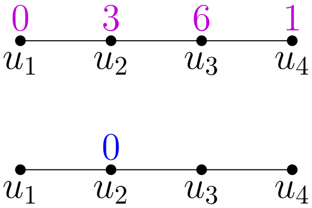

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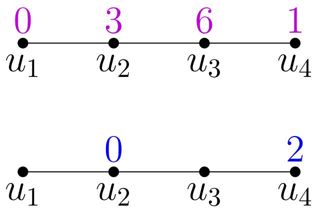

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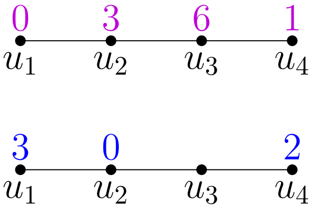

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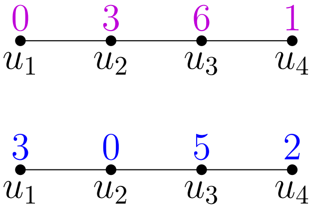

Radio k-chromatic number:

$rc_k(G) = min\{\ell$: G admits an ℓ-radio k-coloring$\}$

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Radio number:

$rn(G) = rc_k(G)$ for $k = diam(G)$.

Power of path

The graph P_n^m on $n+1$ vertices obtained by adding edges between the vertices of P_n that are at most m distance apart.

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Figure: P_{16}^4

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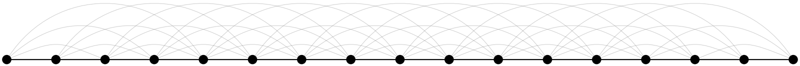
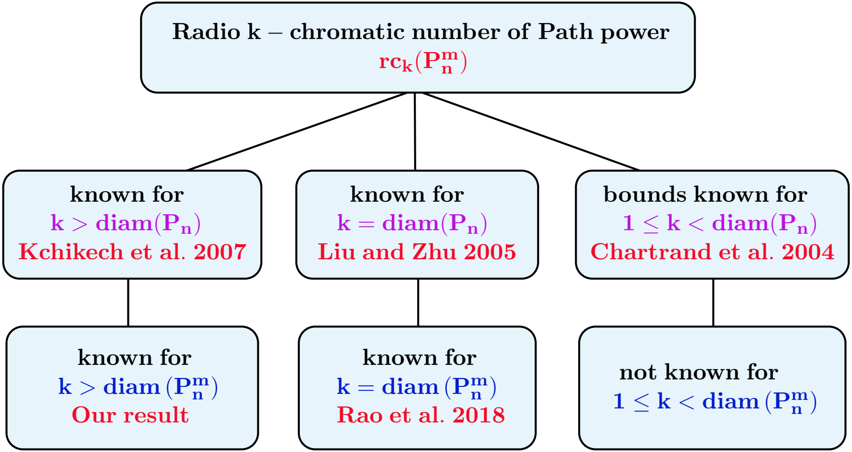


Figure: P_{16}^4


Radio k – chromatic number of Path power
$rc_k(P_n^m)$
known for
$k > diam(P_n)$
Kchikech et al. 2007
known for
$k = diam(P_n)$
Liu and Zhu 2005
bounds known for
$1 \leq k < diam(P_n)$
Chartrand et al. 2004
known for
$k > diam(P_n^m)$
Our result
known for
$k = diam(P_n^m)$
Rao et al. 2018
not known for
$1 \leq k < diam(P_n^m)$

Theorem

For all $k > diam(P_n^m)$, *we have*

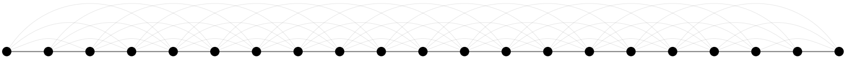
$$rc_k(P_n^m) = \begin{cases} nk - \frac{n^2-m^2}{2m} & \text{if } \lceil \frac{n}{m} \rceil \text{ is odd and } m|n, \\ nk - \frac{n^2-s^2}{2m} + 1 & \text{if } \lceil \frac{n}{m} \rceil \text{ is odd and } m \nmid n, \\ nk - \frac{n^2}{2m} + 1 & \text{if } \lceil \frac{n}{m} \rceil \text{ is even and } m|n, \\ nk - \frac{n^2-(m-s)^2}{2m} + 1 & \text{if } \lceil \frac{n}{m} \rceil \text{ is even and } m \nmid n, \end{cases}$$

where $s \equiv n \pmod{m}$.

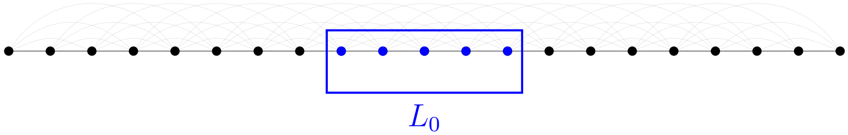
Lower bound

- We define a lower bound technique for $k \geq diam(P_n^m)$.
- This technique is applicable to any graph G in general for the case $k \geq diam(G)$.
- Two cases: diameter odd and diameter even.

Diameter odd



Diameter odd



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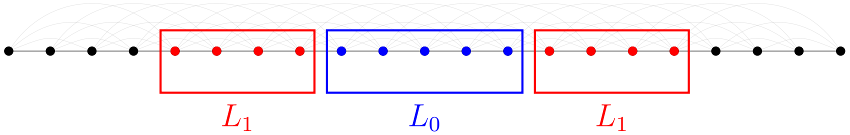

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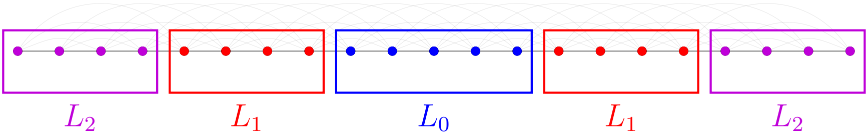

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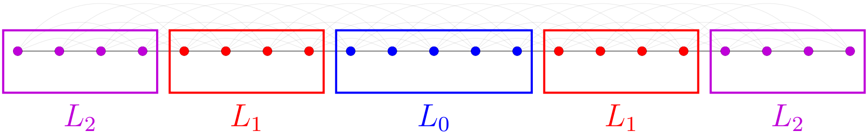

No two vertices receive the same color.

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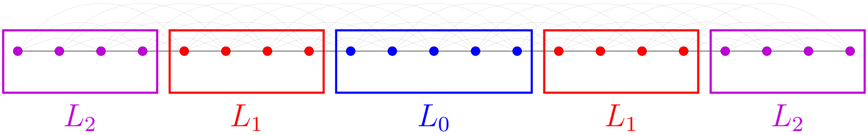

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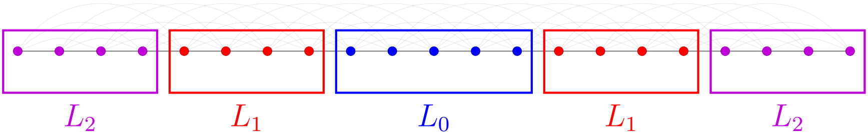
As diameter is less than k, any two vertices are at distance at most $k-1$. So, their color difference must be at least one.

Diameter odd



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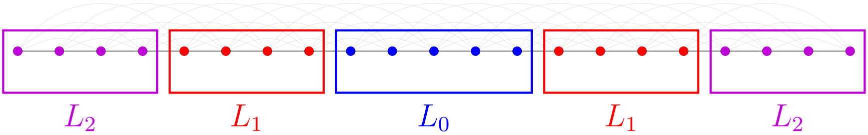

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- The colors can be ordered in an ascending order.
  $c(v_0) < c(v_1) < \cdots < c(v_n)$

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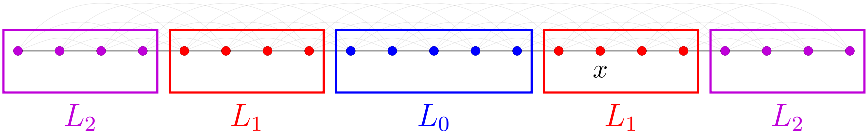

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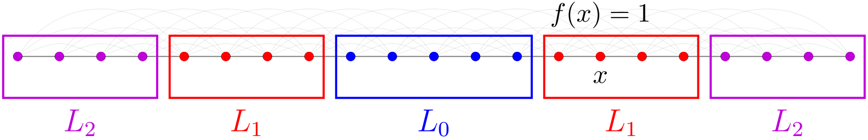

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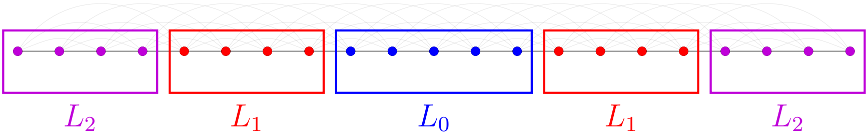

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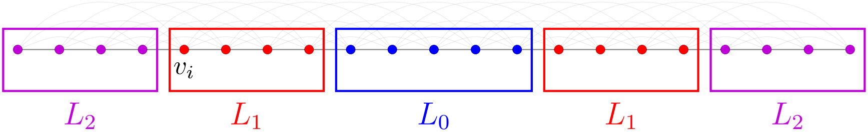

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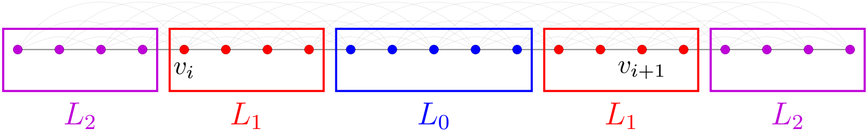

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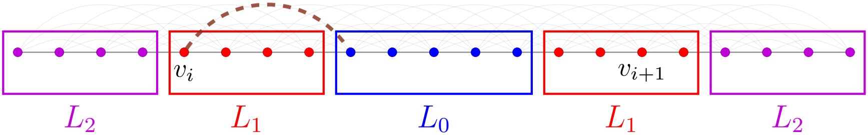

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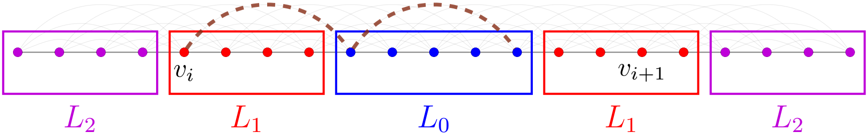

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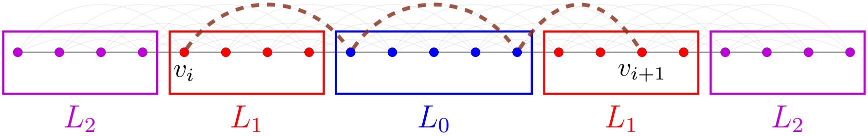

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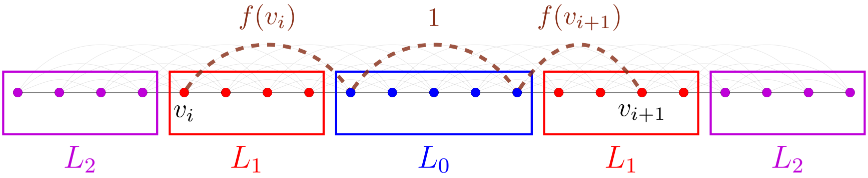

$$d(v_i, v_{i+1}) \leq f(v_i) + 1 + f(v_{i+1}).$$

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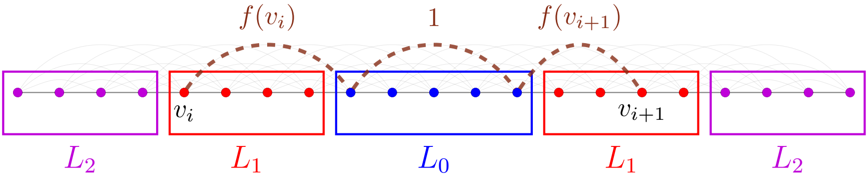

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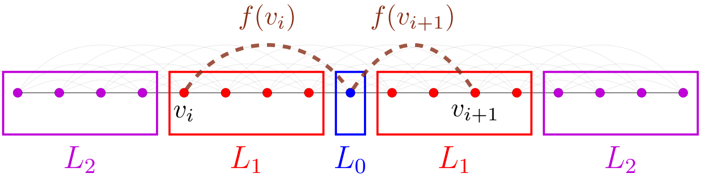

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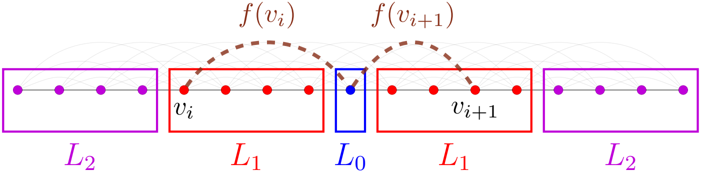

$$d(v_i, v_{i+1}) \leq f(v_i) + f(v_{i+1}).$$

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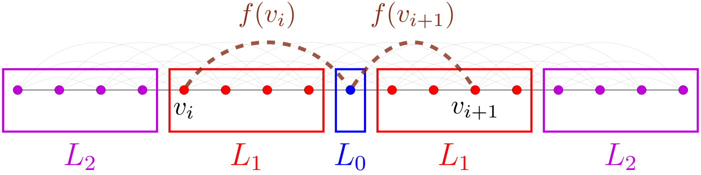

$$d(v_i, v_{i+1}) \leq f(v_i) + f(v_{i+1}).$$

$$c(v_{i+1}) - c(v_i) \geq k + 1 - f(v_i) - f(v_{i+1}).$$

Optimally and loosely colored pair

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$$c(v_{i+1}) - c(v_i) = k - f(v_i) - f(v_{i+1}) + \epsilon.$$

Optimally and loosely colored pair

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Loosely colored pair

$c(v_{i+1}) - c(v_i) > k - f(v_i) - f(v_{i+1}) + \epsilon.$

$$(v_0, v_1, \cdots, v_n) = Y_0 X_1 Y_1 X_2 \cdots X_t Y_t$$

Y_i - loosely colored sequences.
X_j - maximal optimally colored sequences.

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$$\begin{aligned} rc_k(G) &= \sum_{i=0}^{n-1} [\phi(v_{i+1}) - \phi(v_i)] \\ &\geq |S| + \sum_{i=0}^{n-1} [k - f(v_i) - f(v_{i+1}) + \epsilon]. \end{aligned}$$

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$S = \{v_i : (v_i, v_{i+1})$ is loosely colored, where $0 \leq i \leq n-1\}$

Remark

- Lower bound technique can be applied to a graph G of diameter more than k also.
- Take a subgraph H of G induced on $\bigcup_{i=0}^{q} L_i$, where $q = \lfloor \frac{k}{2} \rfloor$ and $k \geq diam(H)$.
- $rc_k(H) \leq rc_k(G)$.

Upper bound

For $k > diam(P_n^m)$,

Case 1: $diam(P_n^m)$ ***even,*** $m|n$.

Case 2: $diam(P_n^m)$ ***even,*** $m \nmid n$.

Case 3: $diam(P_n^m)$ ***odd,*** $m|n$.

Case 4: $diam(P_n^m)$ ***odd,*** $m \nmid n$.

Open problems

- Find $rc_k(P_n^m)$ for $k < diam(P_n^m)$.

- Can we give an upper bound for $rc_k(G)$ in terms of maximum degree $\Delta(G)$?


THANK YOU!