The Interplay of Domination and Separation in Graphs

Dipayan Chakraborty and Annegret K. Wagler

Université Clermont-Auvergne
Mines de Saint-Étienne
Clermont-Auvergne INP
LIMOS, UMR 6158 CNRS
Clermont-Ferrand, France

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OUTLINE

1. IDENTIFICATION PROBLEMS IN GRAPHS
2. ABOUT SEPARATION PROPERTIES
3. THE INTERPLAY OF DOMINATION AND SEPARATION

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OBJECTIVE

Separate any two nodes of a graph by their unique neighborhoods in a suitably chosen (total-)dominating set (= **code**) of the graph.

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Let $G = (V, E)$ be a graph and denote by $N(i)$ (resp. $N[i]$) the open (resp. closed) neighborhood of a node $i \in V$.

DOMINATION PROPERTIES: $C \subseteq V$ IS

- **dominating** if $N[i] \cap C$ are non-empty sets for all $i \in V$
- **total-dominating** if $N(i) \cap C$ are non-empty sets for all $i \in V$

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Separation properties: $C \subseteq V$ is

- **locating** if $N(i) \cap C$ are distinct sets for all $i \in V \setminus C$
- **open-separating** if $N(i) \cap C$ are distinct sets for all $i \in V$
- **closed-separating** if $N[i] \cap C$ are distinct sets for all $i \in V$
- **full-separating** if it is both open- and closed-separating.

The following identification problems have been studied in the literature by combining a domination and a separation property:

	domination (D-set)	total-domination (TD-set)
location (L-set)	locating-dominating codes (**LD**-codes)	locating total-dominating codes (**LTD**-codes)
closed-separation (C-set)	identifying codes (**ID**-codes)	differentiating total-dominating codes (**DTD**-codes)
open-separation (O-set)	open-separating dominating codes (**OD**-codes)	open-locating dominating codes (**OLD**-codes)
full-separation (F-set)	full-separating dominating codes (**FD**-codes)	full-separating total-dominating codes (**FTD**-codes)

Combining domination and separation properties

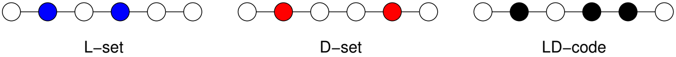
The following identification problems have been studied in the literature by combining a domination and a separation property:

	domination (D-set)	total-domination (TD-set)
location (L-set)	locating dominating codes (**LD**-codes)	locating total-dominating codes (**LTD**-codes)
closed-separation (I-set)	identifying (dominating) codes (**ID**-codes)	identifying total-dominating codes (**ITD**-codes)
open-separation (O-set)	open-separating dominating codes (**OD**-codes)	open-separating total-dominating codes (**OTD**-codes)
full-separation (F-set)	full-separating dominating codes (**FD**-codes)	full-separating total-dominating codes (**FTD**-codes)

SOME EXAMPLES

For a graph $G = (V, E)$, a subset $C \subseteq V$ is

- locating if $N(i) \cap C$ are distinct sets for all $i \in V \setminus C$
- dominating if $N[i] \cap C$ are non-empty sets for all $i \in V$
- a **locating dominating code** if it is locating and dominating



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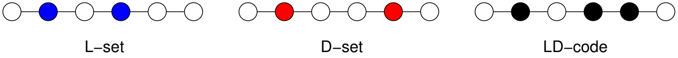
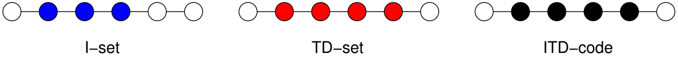


For a graph $G = (V, E)$, a subset $C \subseteq V$ is

- closed-separating if $N[i] \cap C$ are distinct sets for all $i \in V$
- total-dominating if $N(i) \cap C$ are non-empty sets for all $i \in V$
- an **identifying total-dominating code** if it is closed-separating and total-dominating



X-CODE PROBLEM

Given a graph G and $X \in \{LD, OD, ID, FD\} \cup \{LTD, OTD, ITD, FTD\}$,
find an X-code of minimum cardinality $\gamma^X(G)$ in G.

- The related decision problems are all NP-complete:
  - LD (Colbourn, Slater, Steward 1987)
  - ID (Charon, Hudry, Lobstein 2003)
  - OTD (Seo, Slater 2010)
  - LTD (Jayagopal, Miller, Rajan, Rajasingh 2017)
  - ITD (Goyal, Panda, Pradhan 2021)
  - OD (Chakraborty and W. 2024)
  - FD and FTD (Chakraborty and W. 2025)
- Active research area, see e.g. the bibliography by Jean and Lobstein:
  `https://dragazo.github.io/bibdom/main.pdf`

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- But: Separation properties not yet studied independently from codes!

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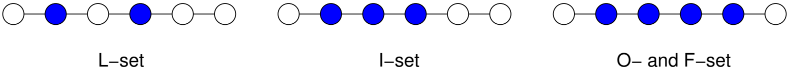
Recall that for a graph $G = (V, E)$, a subset $C \subseteq V$ is

- **locating (L-set)** if $N(i) \cap C$ are distinct sets for all $i \in V \setminus C$
- **closed-separating (I-set)** if $N[i] \cap C$ are distinct sets for all $i \in V$
- **open-separating (O-set)** if $N(i) \cap C$ are distinct sets for all $i \in V$
- **full-separating (F-set)** if it is both open- and closed-separating.



Separation numbers of graphs

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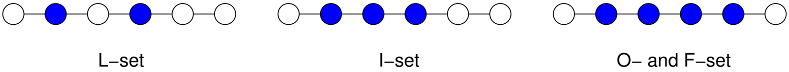
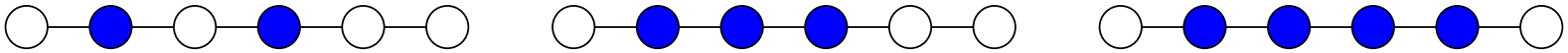


For all separation properties $S \in \{L,I,O,F\}$, the **S-number** $\gamma^{S}(G)$ of G is the cardinality of a minimum S-set in G.

Theorem (Chakraborty and W. 2025)

For a given graph G and an integer ℓ, it is NP-complete to decide whether $\gamma^{S}(G) \leq \ell$ for any $S \in \{L,I,O,F\}$.

Separation and complementation



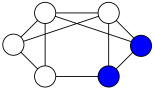
L–set I–set O– and F–set

Concerning the S-numbers of a graph G and its complement $\bar{G}$, we can show:

Theorem (Chakraborty and W. 2025)

For any graph G, we have

- $\gamma^{\mathsf{S}}(G) = \gamma^{\mathsf{S}}(\bar{G})$ *for* $S \in \{L, F\}$*;*
- $\gamma^{\mathsf{O}}(G) = \gamma^{\mathsf{I}}(\bar{G})$*.*



L–set O–set I– and F–set

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THEOREM (CHAKRABORTY AND W. 2025)

For any graph G, we have for SD-codes:

- $\gamma^{\mathsf{SD}}(G) \leq \gamma^{\mathsf{S}}(G) + 1$ *for all* $S \in \{L, O, I, F\}$

Relations of S-sets with SD- and STD-codes

Theorem (Chakraborty and W. 2025)

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Theorem (Chakraborty and W. 2025)

For any graph G, we have for STD-codes:

- $\gamma^{\mathsf{STD}}(G) \leq \gamma^{\mathsf{S}}(G) + 1$ *for* $S \in \{O, F\}$
- $\gamma^{\mathsf{STD}}(G) \leq 2\gamma^{\mathsf{S}}(G)$ *for* $S \in \{L, I\}$

Relations of S-sets with SD- and STD-codes

Theorem (Chakraborty and W. 2025)

For any graph G, we have for SD-codes:

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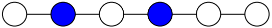
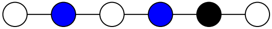
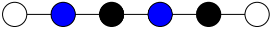


L−set
(no D−set, no TD−set)

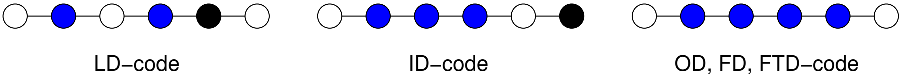


LD−code



LTD−code

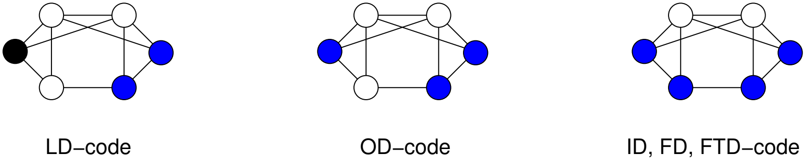
Consequences for complements



Corollary

For any graph G, the following values differ by at most 1:

- $\gamma^{\mathrm{LD}}(G)$ *and* $\gamma^{\mathrm{LD}}(\bar{G})$*;*
- $\gamma^{\mathrm{OD}}(G)$ *and* $\gamma^{\mathrm{ID}}(\bar{G})$*;*
- $\gamma^{\mathrm{FD}}(G)$ *and* $\gamma^{\mathrm{FD}}(\bar{G})$ *as well as* $\gamma^{\mathrm{FTD}}(G)$ *and* $\gamma^{\mathrm{FTD}}(\bar{G})$*.*



Conclusions and conjectures

The previous results show that separation is almost domination due to

$$\gamma^{\mathsf{S}}(G) \le \gamma^{\mathsf{SD}}(G) \le \gamma^{\mathsf{S}}(G) + 1 \; \forall S \in \{L, I, O, F\},$$

and that open-separation is almost total-domination by

$$\gamma^{\mathsf{S}}(G) \le \gamma^{\mathsf{SD}}(G) \le \gamma^{\mathsf{STD}}(G) \le \gamma^{\mathsf{S}}(G) + 1 \text{ for } S \in \{O, F\}.$$

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It is known that

- $\gamma^{\mathsf{OD}}(G)$ and $\gamma^{\mathsf{OTD}}(G)$ (Chakraborty and W. 2024)
- $\gamma^{\mathsf{FD}}(G)$ and $\gamma^{\mathsf{FTD}}(G)$ (Chakraborty and W. 2025)

differ by at most 1, but that it is NP-hard to decide whether the two values are equal.

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differ by at most 1, but that it is NP-hard to decide whether the two values are equal.

Conjecture

It is also NP-hard to decide whether the following pairs of values are equal:

- $\gamma^{\mathsf{S}}(G)$ and $\gamma^{\mathsf{SD}}(G)$ for $S \in \{L, I\}$,
- $\gamma^{\mathsf{SD}}(G)$ and $\gamma^{\mathsf{SD}}(\bar{G})$ for $S \in \{L, F\}$,
- $\gamma^{\mathsf{ID}}(G)$ and $\gamma^{\mathsf{OD}}(\bar{G})$.

37th International Workshop on Combinatorial Algorithms

Hope to see you June 8-11, 2026 in Clermont-Ferrand, France!