# On clique numbers of colored mixed graphs

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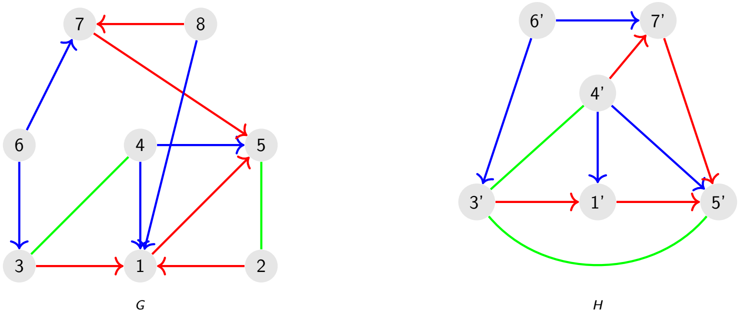
# Discussion about...

1. $(m, n)$-graphs
2. homomorphisms between $(m, n)$-graphs
3. $(m, n)$-relative clique numbers
4. $(m, n)$-absolute clique numbers
5. $(m, n)$-chromatic numbers
6. parameters on different graph families

# $(m,n)$-graphs: $m$ types of arcs and $n$ types of edges




(2, 1)-Graphs

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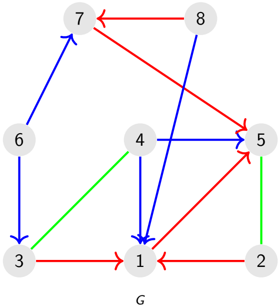


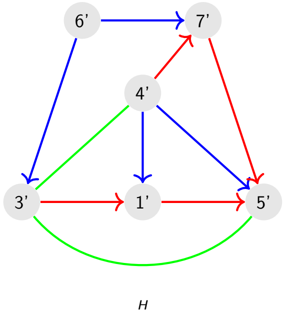


$(2,1)$-Graphs

$(0,1)$-graphs $=$ normal graphs

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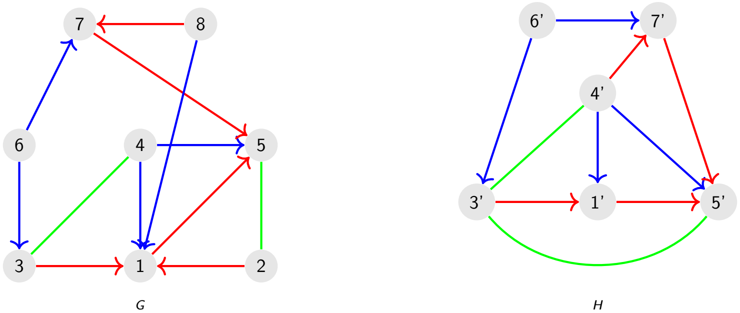
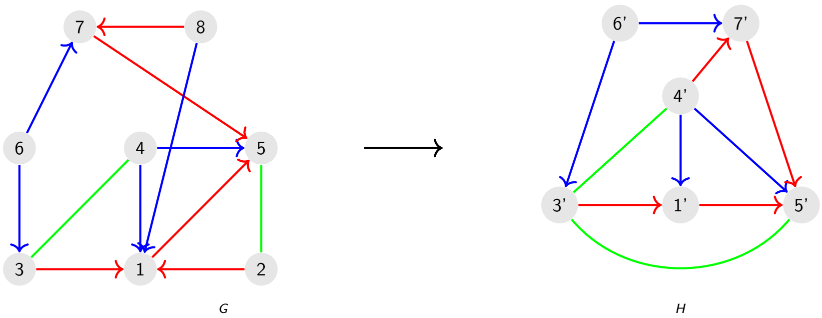



$(2, 1)$-Graphs

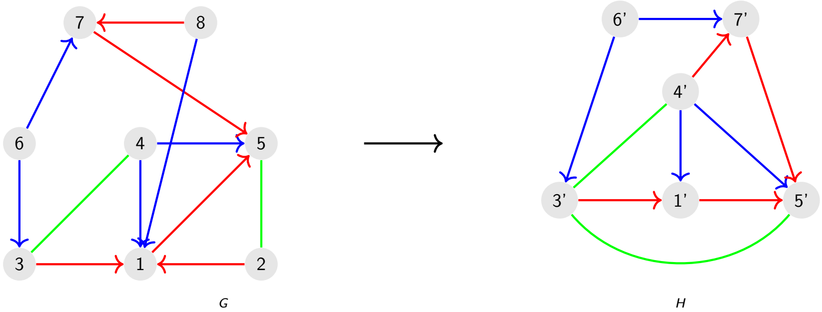
$(0, 1)$-graphs $=$ normal graphs
$(1, 0)$-graphs $=$ oriented graphs

# Homomorphism: adjacency, type and direction preserving vertex mapping




$f : G \to H$. $f(2) = 3'$, $f(8) = 4'$ and $f(i) = i'$ for all $i \neq 2, 8$

# Morphism in motion!

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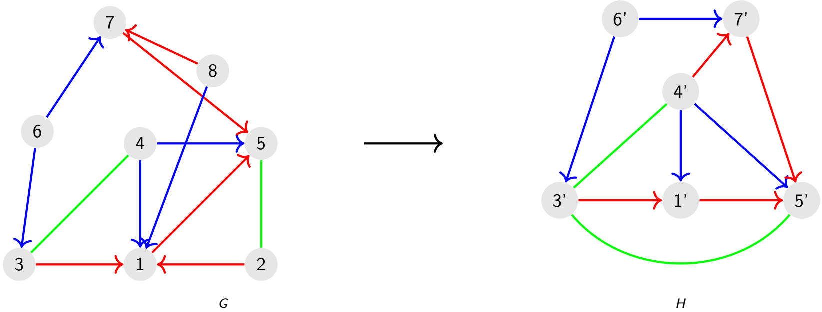


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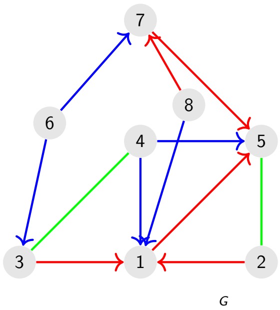


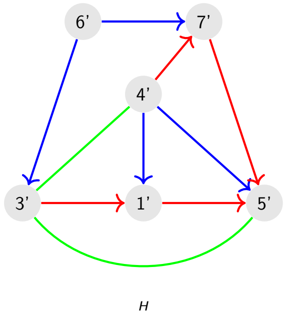


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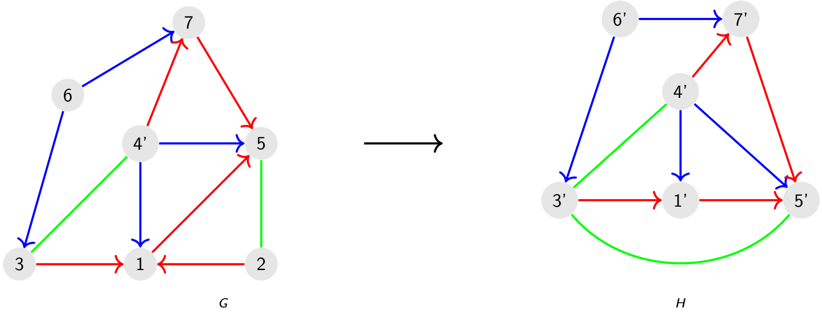


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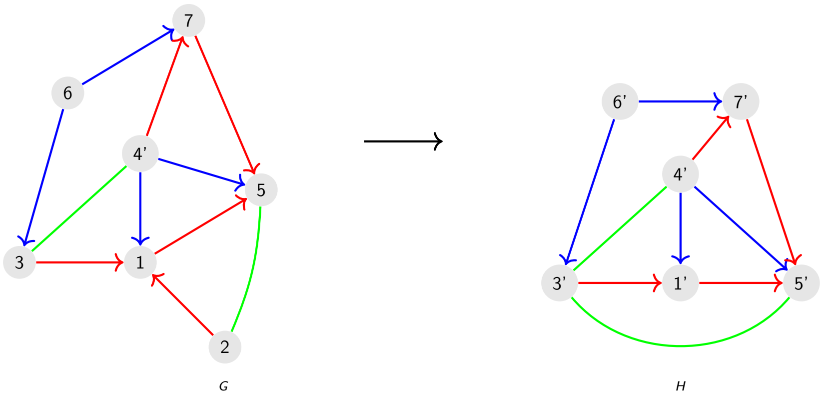


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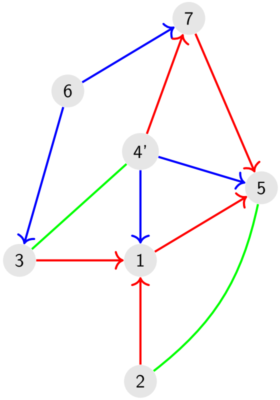
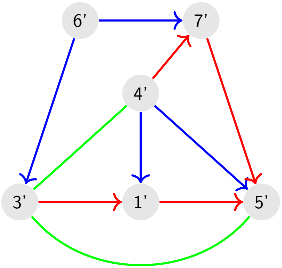



$\longrightarrow$




$G$ $H$

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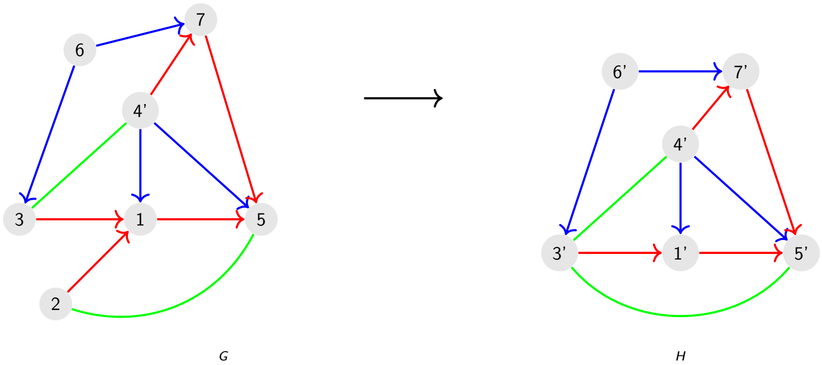


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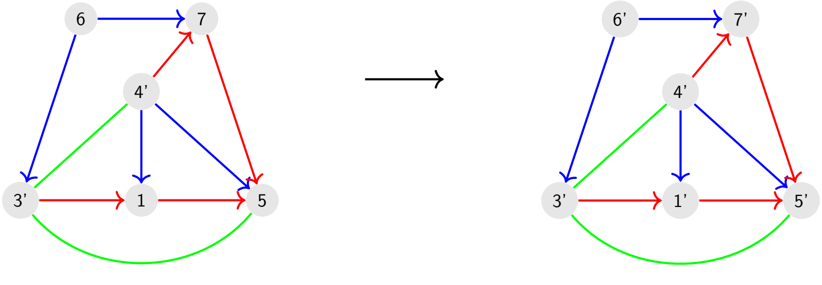


*G* *H*

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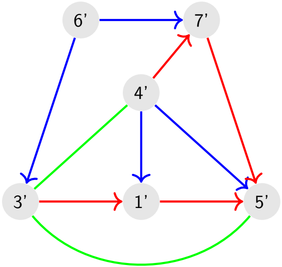
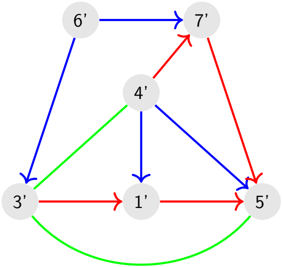



*G*

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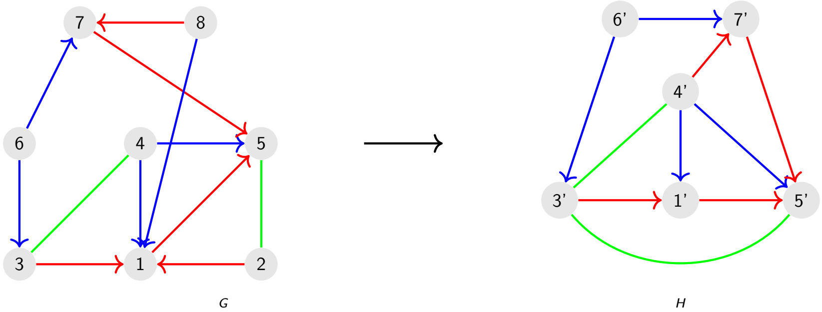


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# The special 2-path characterization


Edge / arc

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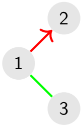
Edge / arc


Loop




Different types

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Edge / arc


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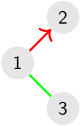


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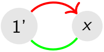


Type mismatch

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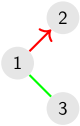
Loop

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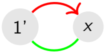


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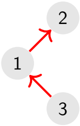


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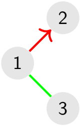


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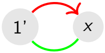


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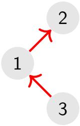


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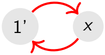
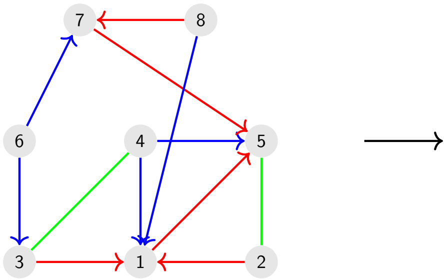
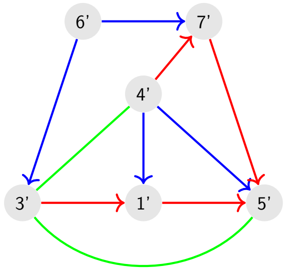



Direction mismatch

# $\chi_{m,n}(G)$: $\min |V(H)|, \ \forall \ G \to G^*$




$G: \ \chi_{m,n}(G) = 6 \neq 8 = |V(G)|$

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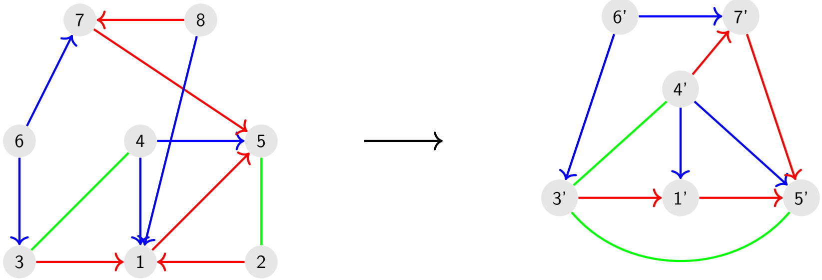


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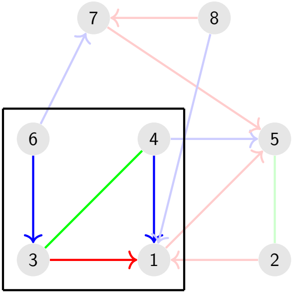
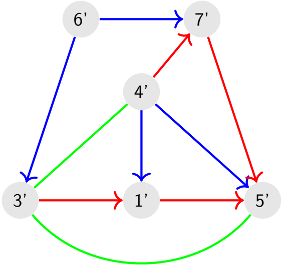



Absolute clique= $\{1, 3, 4, 6\}$




*H* absolute clique

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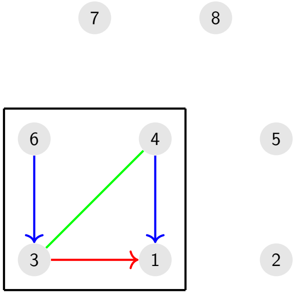
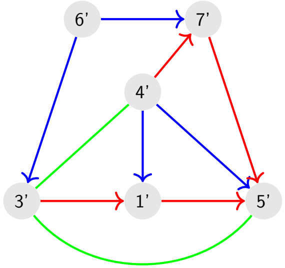



Absolute clique= $\{1, 3, 4, 6\}$. $\omega_{a(m.n)}(G) = 4$




$\omega_{a(m,n)}(H) = \chi_{m,n}(H) = 6$

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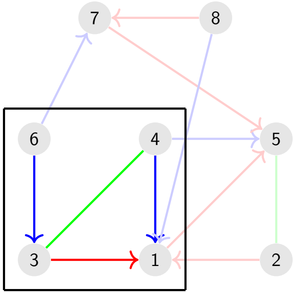


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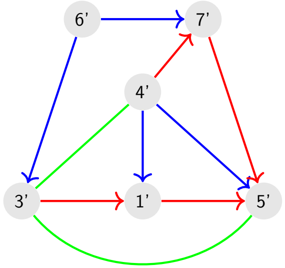


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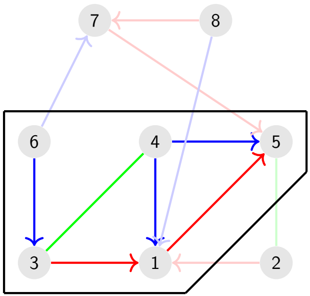


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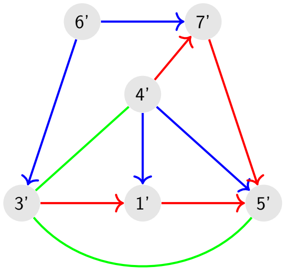


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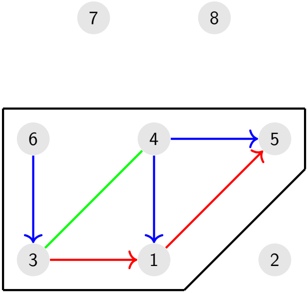


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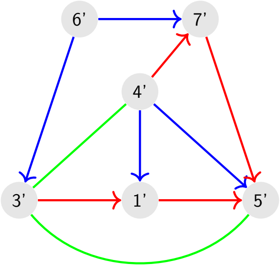


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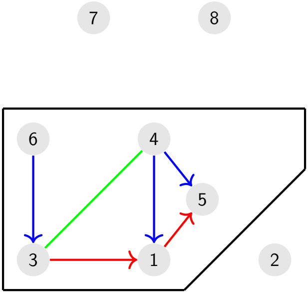


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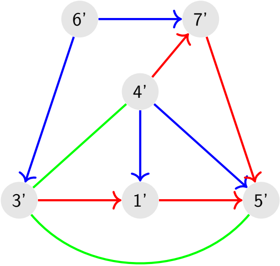


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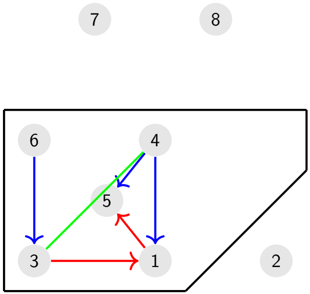


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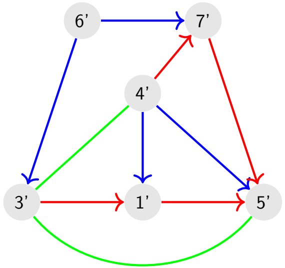


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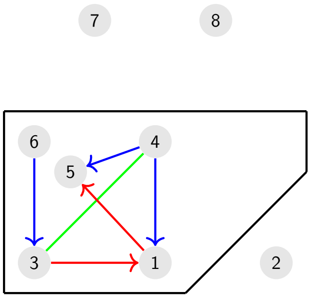


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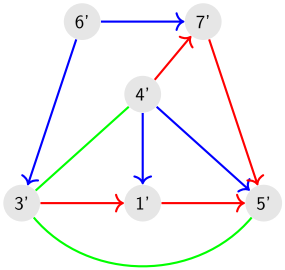


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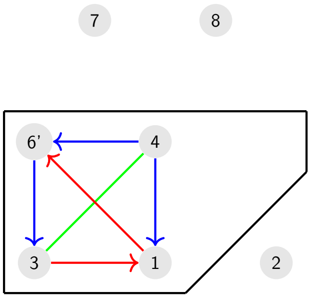


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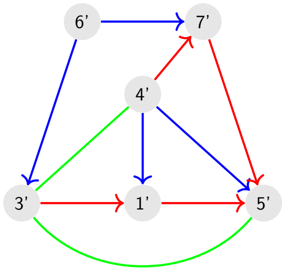


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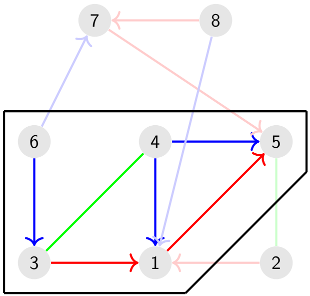


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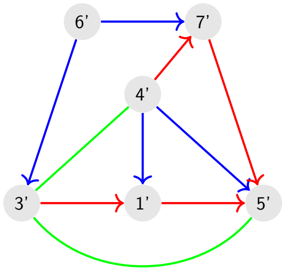


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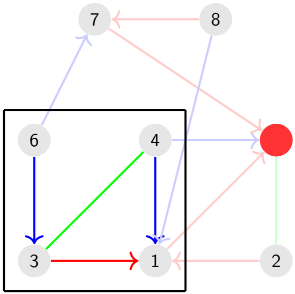


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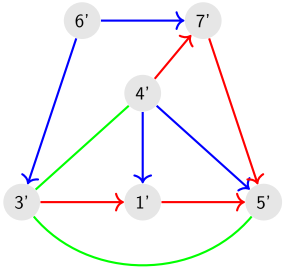


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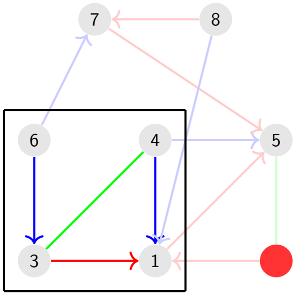


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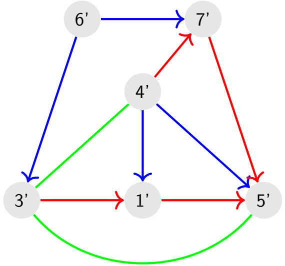
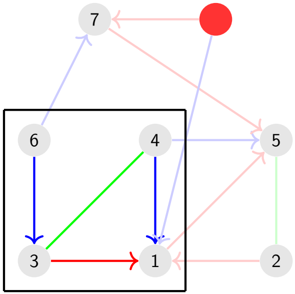



$\omega_{a(m,n)}(H) = \chi_{m,n}(H) = 6$

# $(m, n)$-absolute clique number $\omega_{a(m,n)}(G)$
# max $|V(C)|$ in $G$, $C$ an absolute clique

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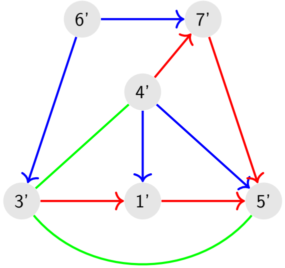


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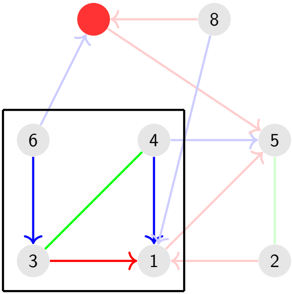


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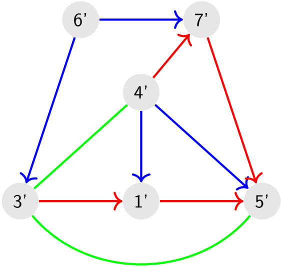


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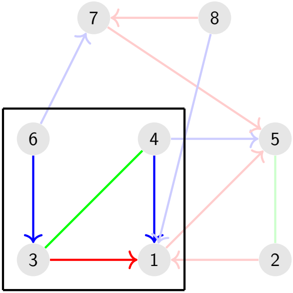


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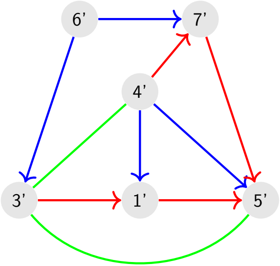


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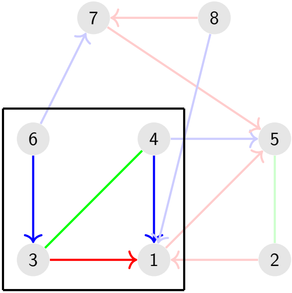


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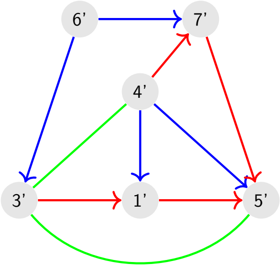


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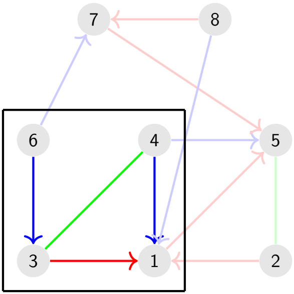


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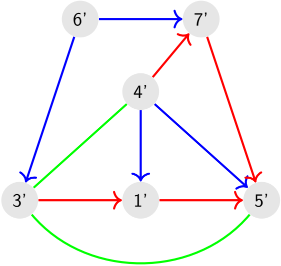


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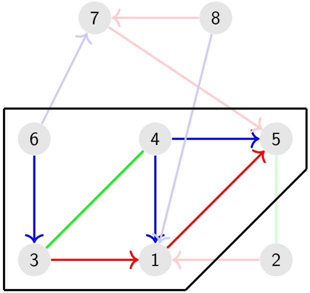


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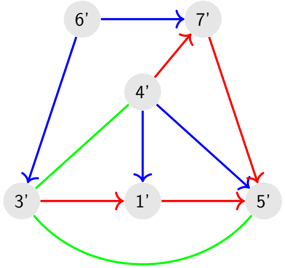


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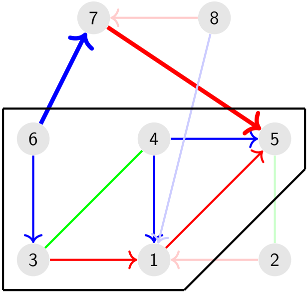


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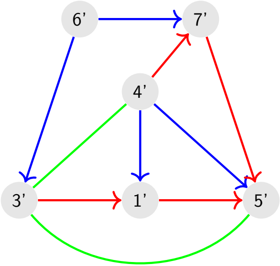


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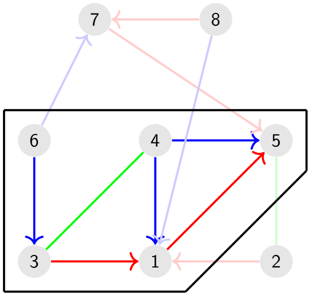
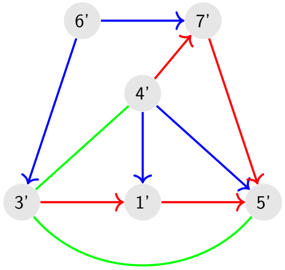



Relative Clique= $\{1, 3, 4, 5, 6\}$




$H$ relative clique

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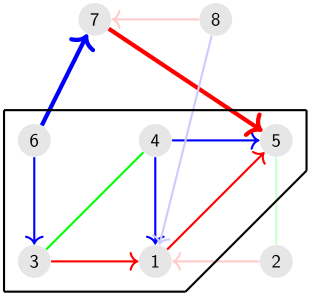


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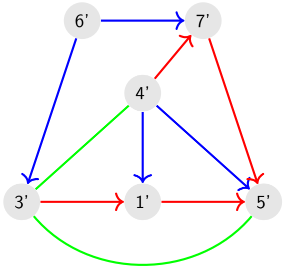


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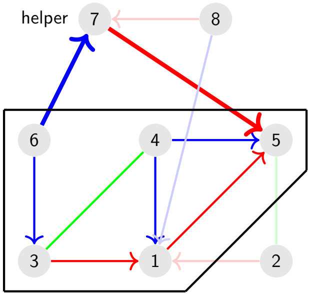


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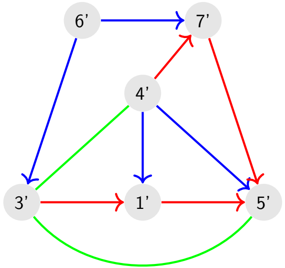


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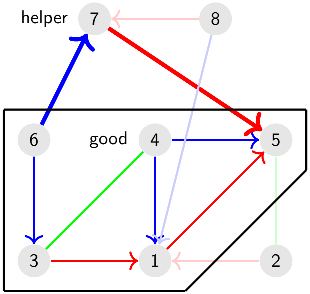


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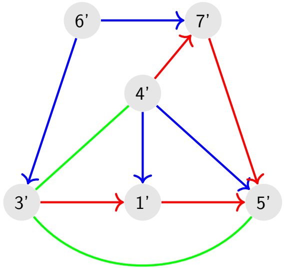
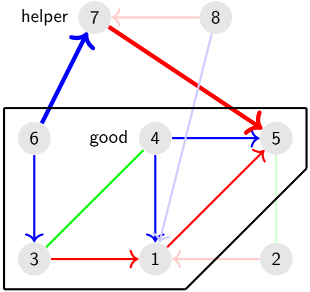
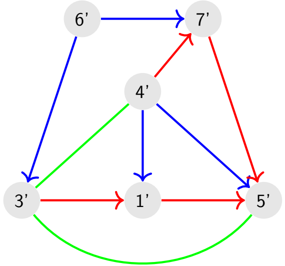



$H$ relative clique

# $(m, n)$-relative clique number $\omega_{r(m,n)}(G)$
# max $|R|$ in $G$, $R$ relative clique

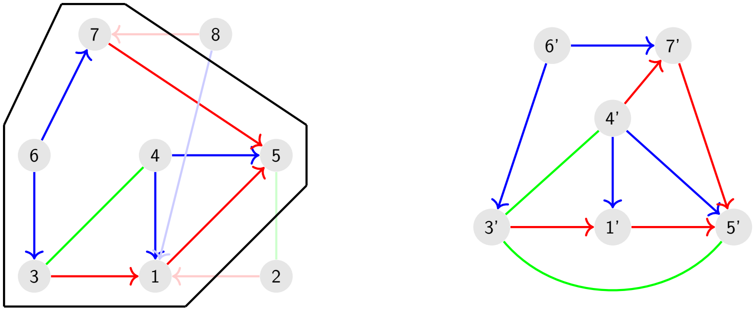



$G$: $\omega_{r(m,n)}(G) = 5$




$H$ : $\omega_{r(m,n)}(H) = 6$

# $(m, n)$-relative clique number $\omega_{r(m,n)}(G)$
# max $|R|$ in $G$, $R$ relative clique

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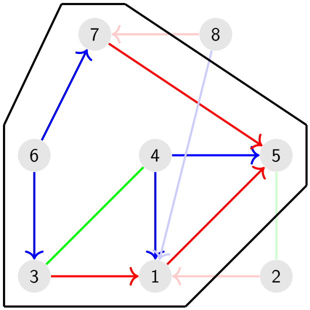


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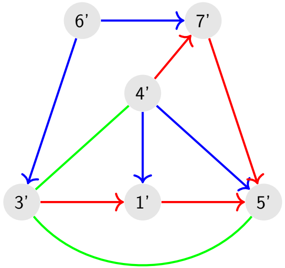


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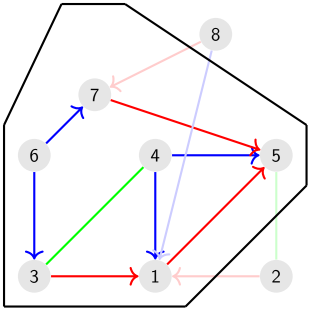


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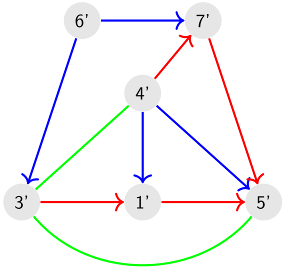


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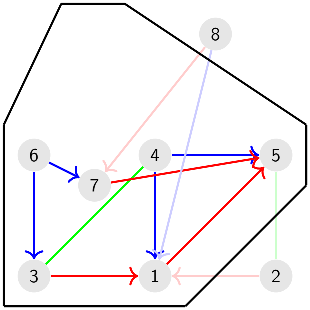


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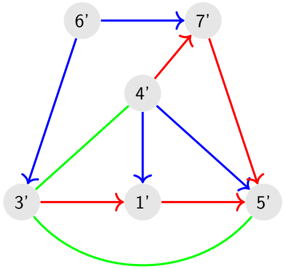


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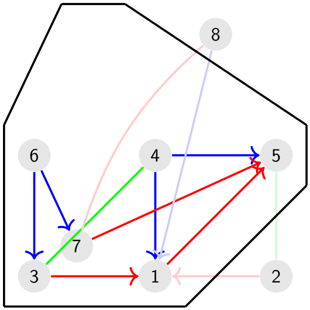


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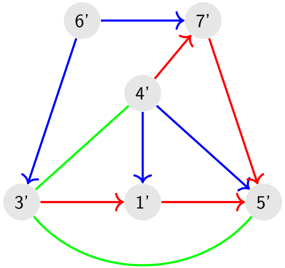


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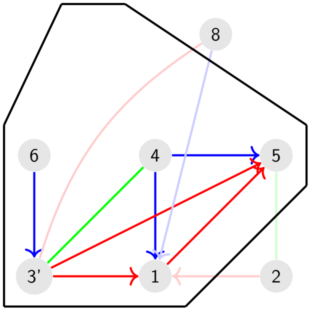


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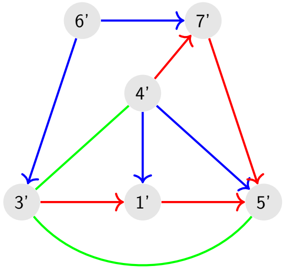


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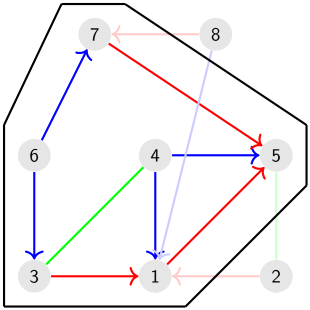


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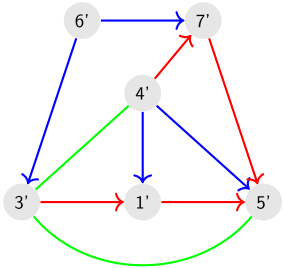


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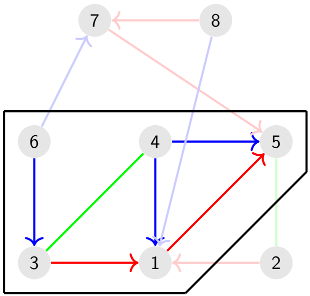


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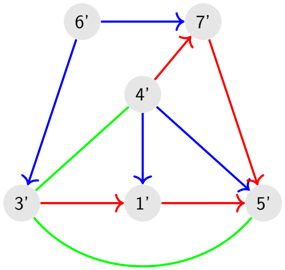


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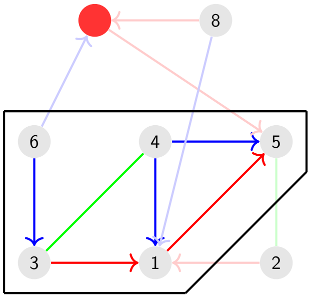


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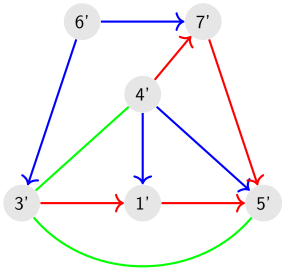


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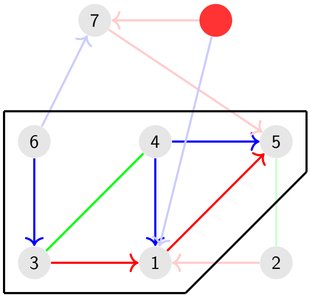


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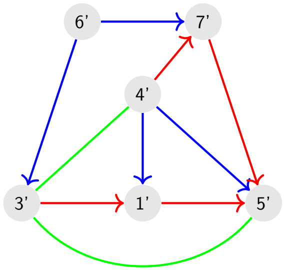


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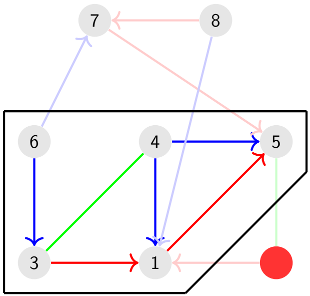


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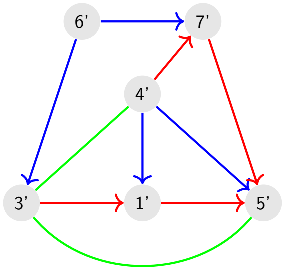


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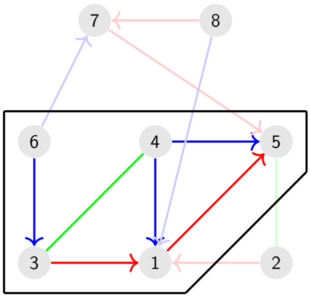


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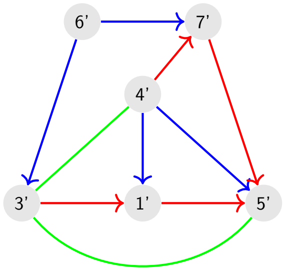
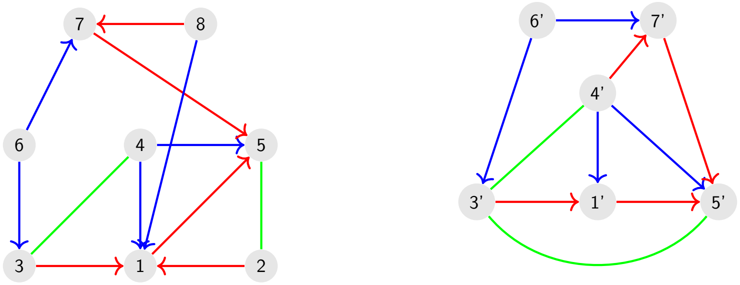



$H$ : $\omega_{r(m,n)}(H) = 6$

# Parameter comparison




$\omega_{a(m,n)}(G) = 4$
$\omega_{r(m,n)}(G) = 5$
$\chi_{m,n}(G) = 6$

$\omega_{a(m,n)}(H) = 6$
$\omega_{r(m,n)}(H) = 6$
$\chi_{m,n}(H) = 6$

# In general...

$$\omega_{a(m,n)}(G) \leq \omega_{r(m,n)}(G) \leq \chi_{m,n}(G)$$

# Parameter for family $\mathcal{F}$

- $\chi_{m,n}(\mathcal{F}) = \max \chi_{m,n}(G),\ G \in \mathcal{F}$
- $\omega_{a(m,n)}(\mathcal{F}) = \max \omega_{a(m,n)}(G),\ G \in \mathcal{F}$
- $\omega_{r(m,n)}(\mathcal{F}) = \max \omega_{r(m,n)}(G),\ G \in \mathcal{F}$

# Bensmail, Duffy and Sen:

**Theorem**

*Let $\mathcal{O}$ denote the class of all outerplanar graphs. Then, for all $(m, n) \neq (0, 1)$, we have,*

$$\omega_{a(m,n)}(\mathcal{O}) = \omega_{r(m,n)}(\mathcal{O}) = 3p + 1.$$

**Theorem**

*Let $\mathcal{P}$ denote the class of all planar graphs. Then, for all $(m, n) \neq (0, 1)$, we have,*

$$3p^2 + p + 1 \leq \omega_{a(m,n)}(\mathcal{P}) \leq 9p^2 + 2p + 2.$$

# Our work

Clique numbers for:

1. Partial 2-trees of girth $g$: $\mathcal{T}_g^2$
2. Planar graphs of girth $g$: $\mathcal{P}_g$
3. Graphs of maximum degree $\Delta$: $\mathcal{G}_\Delta$

$p = 2m + n$

$\nu(2, \Delta)$: max $|G|$, $diam(G) = 2$ and max $\deg(G) = \Delta$

# $\mathcal{T}_g^2$: Partial 2-trees of girth $g$

1. $\omega_{a(m,n)}(\mathcal{T}_3^2) = \omega_{r(m,n)}(\mathcal{T}_3^2) = p^2 + p + 1$.
2. $\omega_{a(m,n)}(\mathcal{T}_4^2) = \omega_{r(m,n)}(\mathcal{T}_4^2) = p^2 + 2$.
3. $\omega_{a(m,n)}(\mathcal{T}_5^2) = \omega_{r(m,n)}(\mathcal{T}_5^2) = \max(p + 1, 5)$ for $(m, n) \neq (0, 2)$.
4. $\omega_{a(0,2)}(\mathcal{T}_5^2) = 3$ and $\omega_{r(0,2)}(\mathcal{T}_5^2) = 4$.
5. $\omega_{a(m,n)}(\mathcal{T}_g^2) = \omega_{r(m,n)}(\mathcal{T}_g^2) = p + 1$ for all $g \geq 6$.

## $\mathcal{P}_g$: Planar of girth $g$

1. $3p^2 + p + 1 \leq \omega_{a(m,n)}(\mathcal{P}_3) \leq \omega_{r(m,n)}(\mathcal{P}_3) \leq 42p^2 - 11.$
2. $p^2 + 2 = \omega_{a(m,n)}(\mathcal{P}_4) \leq \omega_{r(m,n)}(\mathcal{P}_4) \leq 14p^2 + 1.$
3. $\max(p+1, 5) = \omega_{a(m,n)}(\mathcal{P}_5) \leq \omega_{r(m,n)}(\mathcal{P}_5) = \max(p+1, 6).$
4. $p + 1 = \omega_{a(m,n)}(\mathcal{P}_6) \leq \omega_{r(m,n)}(\mathcal{P}_6) = \max(p+1, 4).$
5. $\omega_{a(m,n)}(\mathcal{P}_g) = \omega_{r(m,n)}(\mathcal{P}_g) = p + 1$ for $g \geq 7$.

## $\mathcal{G}_\Delta$: Graphs of maximum degree $\Delta$

(i) $\nu(2,\Delta) = \omega_{a(m,n)}(\mathcal{G}_\Delta) \leq \omega_{r(m,n)}(\mathcal{G}_\Delta) \leq \Delta^2 + 1$ for all $\Delta < p$.

(ii) $\omega_{a(m,n)}(\mathcal{G}_\Delta) \leq \omega_{r(m,n)}(\mathcal{G}_\Delta) \leq \Delta^2 + 1$ for all $\Delta = p$.

(iii) $\omega_{a(m,n)}(\mathcal{G}_\Delta) \leq \omega_{r(m,n)}(\mathcal{G}_\Delta) \leq \lfloor \frac{p-1}{p}\Delta^2 \rfloor + \Delta + 1$ for all $\Delta > p$.

(i) $\omega_{a(1,0)}(\mathcal{G}_3) = \omega_{r(1,0)}(\mathcal{G}_3) = 7$ (Das, Prabhu and Sen).

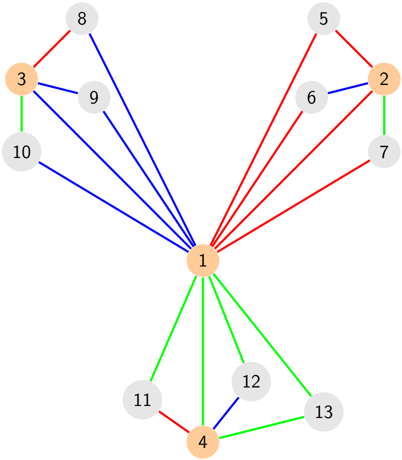
(ii) $\omega_{a(0,n)}(\mathcal{G}_3) = \omega_{r(0,n)}(\mathcal{G}_3) = 8$ for $n = 2, 3$.

(iii) $\omega_{a(m,n)}(\mathcal{G}_3) = \omega_{r(m,n)}(\mathcal{G}_3) = 10$ for $(m,n) = (1,1)$ and $(m,n)$ such that $p \geq 4$.

# Sketch of poof: $\mathcal{T}_g^2$

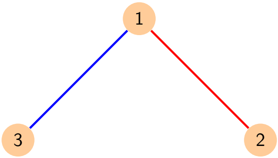
$$\omega_{a(m,n)}(\mathcal{T}_3^2) = \omega_{r(m,n)}(\mathcal{T}_3^2) = p^2 + p + 1.$$

# $\mathcal{T}_3^2$: Partial 2-trees
# Sketch of proof: The lower bound $(p^2 + p + 1)$

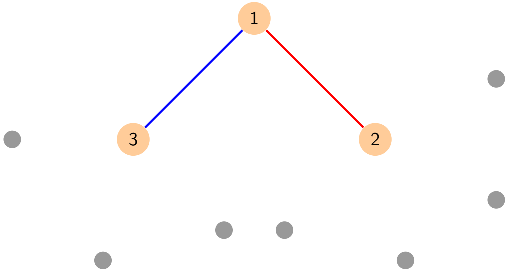



$G$: $(0, 3)$-partial 2-tree. $p = 3$. $\omega_{a(0,3)}(G) = 3^2 + 3 + 1 = 13$

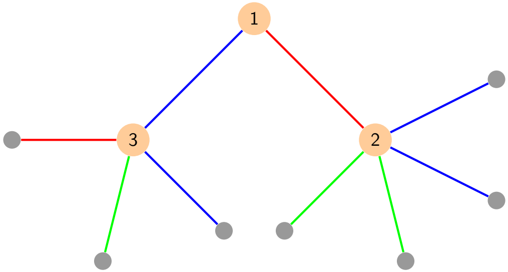
# $\mathcal{T}_3^2$: Partial 2-trees
## Sketch of proof: The upper bound $(p^2 + p + 1)$



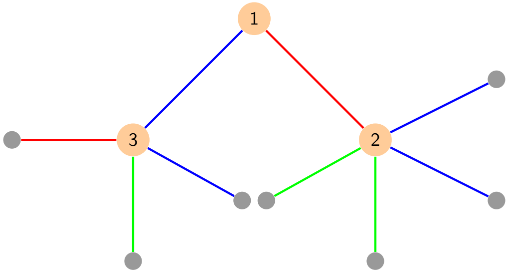
# $\mathcal{T}_3^2$: Partial 2-trees
# Sketch of proof: The upper bound $(p^2 + p + 1)$



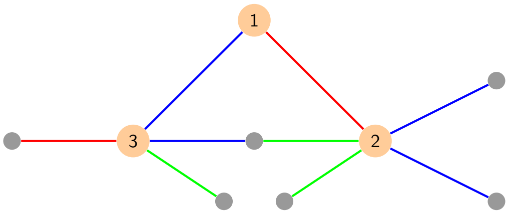
# $\mathcal{T}_3^2$: Partial 2-trees
# Sketch of proof: The upper bound $(p^2 + p + 1)$



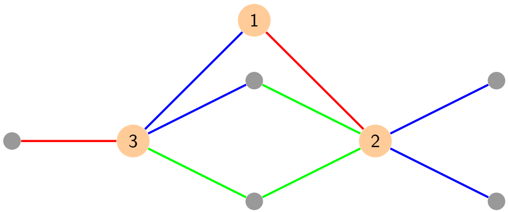
# $\mathcal{T}_3^2$: Partial 2-trees
# Sketch of proof: The upper bound $(p^2 + p + 1)$



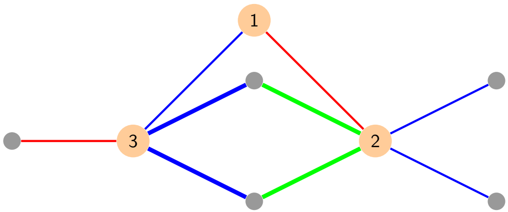
# $\mathcal{T}_3^2$: Partial 2-trees
# Sketch of proof: The upper bound $(p^2 + p + 1)$



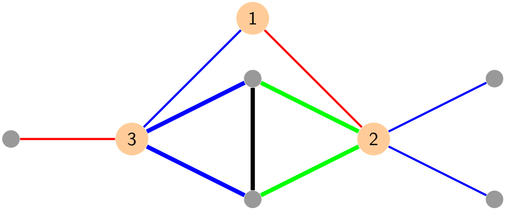
## $\mathcal{T}_3^2$: Partial 2-trees
## Sketch of proof: The upper bound $(p^2 + p + 1)$



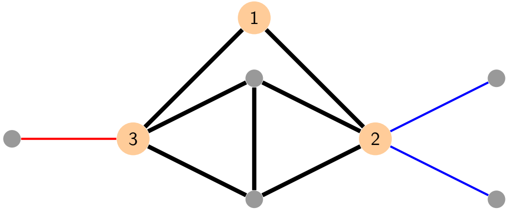
# $\mathcal{T}_3^2$: Partial 2-trees
# Sketch of proof: The upper bound $(p^2 + p + 1)$



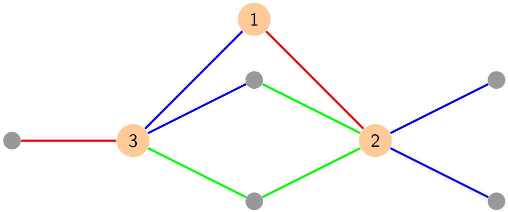
# $\mathcal{T}_3^2$: Partial 2-trees
# Sketch of proof: The upper bound $(p^2 + p + 1)$



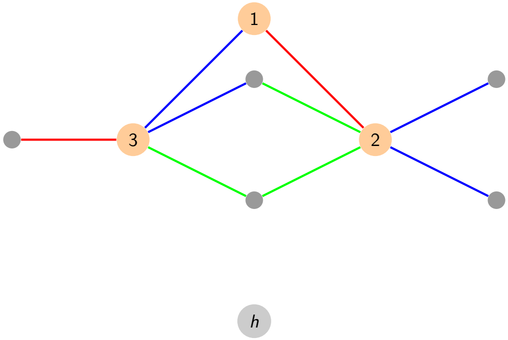
# $\mathcal{T}_3^2$: Partial 2-trees
# Sketch of proof: The upper bound $(p^2 + p + 1)$



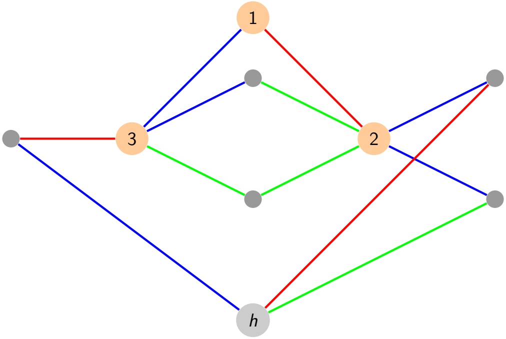
# $\mathcal{T}_3^2$: Partial 2-trees
# Sketch of proof: The upper bound $(p^2 + p + 1)$



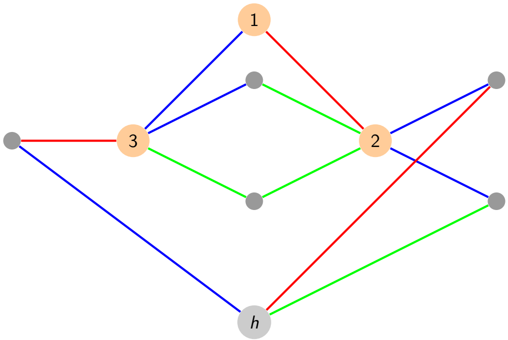
# $\mathcal{T}_3^2$: Partial 2-trees
# Sketch of proof: The upper bound $(p^2 + p + 1)$



# $\mathcal{T}_3^2$: Partial 2-trees
# Sketch of proof: The upper bound $(p^2 + p + 1)$

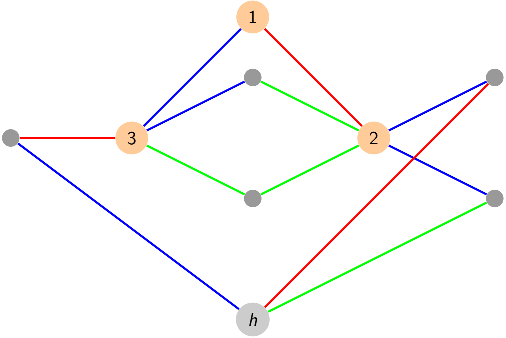


# $\mathcal{T}_3^2$: Partial 2-trees
# Sketch of proof: The upper bound $(p^2 + p + 1)$



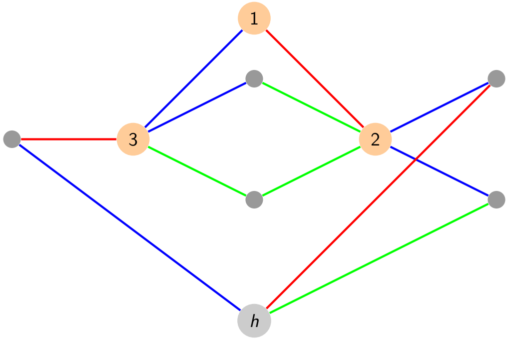

$p_1$ (middle right) $= 2$.

# $\mathcal{T}_3^2$: Partial 2-trees
# Sketch of proof: The upper bound $(p^2 + p + 1)$



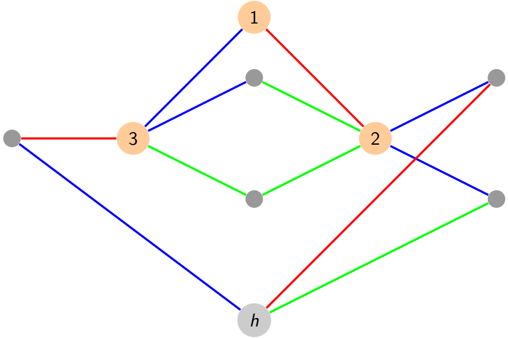

$p_1$ (middle right) $= 2$. $p_2$ (middle left) $= 2$

# $\mathcal{T}_3^2$: Partial 2-trees
# Sketch of proof: The upper bound $(p^2 + p + 1)$



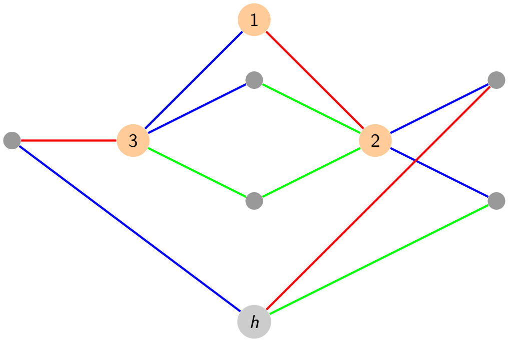

$p_1$ (middle right) $= 2$. $p_2$ (middle left) $= 2$
$q_1$ (right to $h$) $= 2$.

## $\mathcal{T}_3^2$: Partial 2-trees
## Sketch of proof: The upper bound $(p^2 + p + 1)$




$p_1$ (middle right) $= 2$. $p_2$ (middle left) $= 2$
$q_1$ (right to $h$) $= 2$. $q_2$ (left to $h$) $= 1$

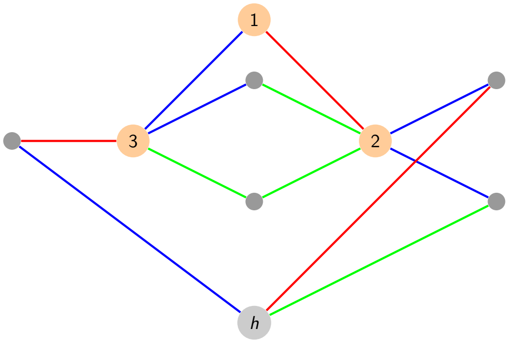
# $\mathcal{T}_3^2$: Partial 2-trees
# Sketch of proof: The upper bound $(p^2 + p + 1)$




$p_1$ (middle right) $= 2$. $p_2$ (middle left) $= 2$
$q_1$ (right to $h$) $= 2$. $q_2$ (left to $h$) $= 1$

$|R| \leq$

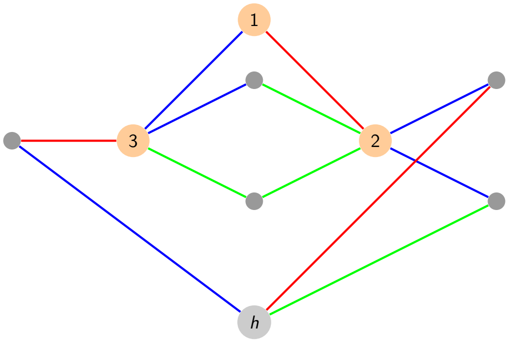
# $\mathcal{T}_3^2$: Partial 2-trees
# Sketch of proof: The upper bound $(p^2 + p + 1)$




$p_1$ (middle right) $= 2$. $p_2$ (middle left) $= 2$
$q_1$ (right to $h$) $= 2$. $q_2$ (left to $h$) $= 1$

$|R| \leq p_1 p_2$

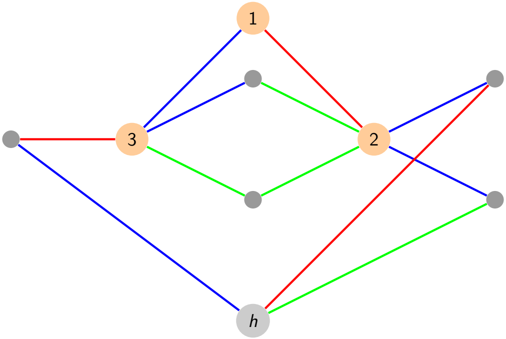
# $\mathcal{T}_3^2$: Partial 2-trees
# Sketch of proof: The upper bound $(p^2 + p + 1)$




$p_1$ (middle right) $= 2$. $p_2$ (middle left) $= 2$
$q_1$ (right to $h$) $= 2$. $q_2$ (left to $h$) $= 1$

$|R| \leq p_1 p_2 + (p - p_1) q_1$

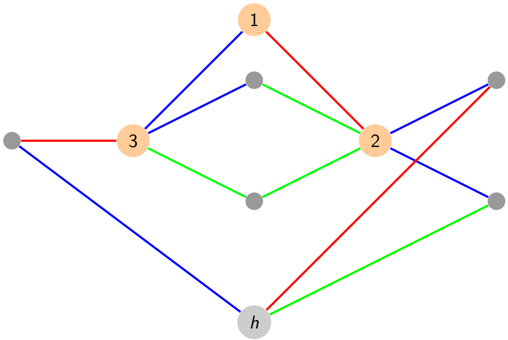
# $\mathcal{T}_3^2$: Partial 2-trees
# Sketch of proof: The upper bound $(p^2 + p + 1)$




$p_1$ (middle right) $= 2$. $p_2$ (middle left) $= 2$
$q_1$ (right to $h$) $= 2$. $q_2$ (left to $h$) $= 1$

$$|R| \leq p_1 p_2 + (p - p_1)q_1 + (p - p_2)q_2$$

## $\mathcal{T}_3^2$: Partial 2-trees
## Sketch of proof: The upper bound $(p^2 + p + 1)$



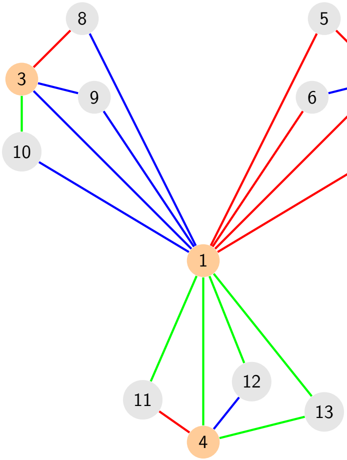

$p_1$ (middle right) $= 2$. $p_2$ (middle left) $= 2$
$q_1$ (right to $h$) $= 2$. $q_2$ (left to $h$) $= 1$

$$|R| \leq p_1 p_2 + (p - p_1) q_1 + (p - p_2) q_2 = (2 \times 2) + (1 \times 2) + (1 \times 1) \leq 7 \leq 13.$$

# Sketch of proof: $\mathcal{T}_g^2$

$$\omega_{a(m,n)}(\mathcal{T}_4^2) = \omega_{r(m,n)}(\mathcal{T}_4^2) = p^2 + 2.$$

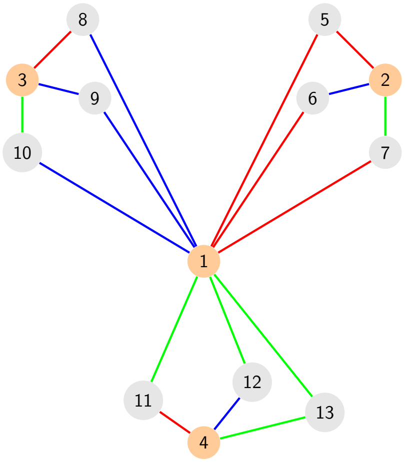
# $\mathcal{T}_4^2$: Triangle-free
# Sketch of proof: $\omega_{r(m,n)}(\mathcal{T}_4^2) = p^2 + 2$




$\mathcal{T}_3^2 : p^2 + p + 1$

$G$: $(0,3)$-partial 2-tree. $p = 3$. $\omega_{a(0,3)}(G) = 3^2 + 3 + 1 = 13$

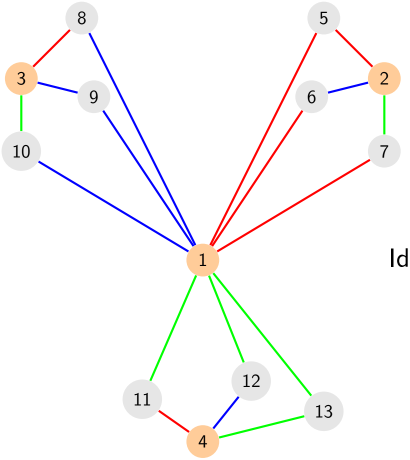
# $\mathcal{T}_4^2$: Triangle-free
# Sketch of proof: $\omega_{r(m,n)}(\mathcal{T}_4^2) = p^2 + 2$




$\mathcal{T}_3^2 : p^2 + p + 1$

$G$: $(0, 3)$-triangle-free partial 2-tree. $p = 3$

# $\mathcal{T}_4^2$: Triangle-free
# Sketch of proof: $\omega_{r(m,n)}(\mathcal{T}_4^2) = p^2 + 2$

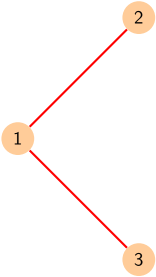



Identify all ORANGE STAR vertices!

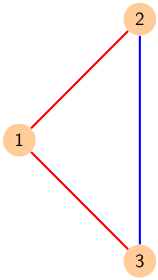
# Sketch of proof: $\mathcal{T}_g^2$, $g \geq 5$ (higher girths)

$$\omega_{a(m,n)}(\mathcal{T}_g^2) = \omega_{r(m,n)}(\mathcal{T}_g^2) = p + 1, \text{ for all } p \geq 4$$

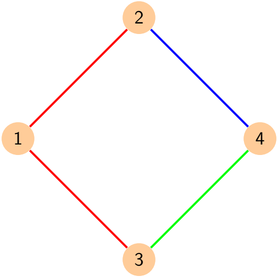
# $\mathcal{T}_g^2$, $g \geq 5$: Higher girths
# Sketch of proof: $\omega_{r(m,n)}(\mathcal{T}_g^2) = p + 1$



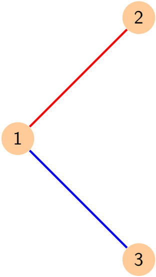
## $\mathcal{T}_g^2$, $g \geq 5$: Higher girths
## Sketch of proof: $\omega_{r(m,n)}(\mathcal{T}_g^2) = p + 1$



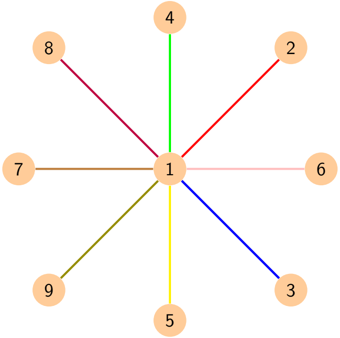
# $\mathcal{T}_g^2$, $g \geq 5$: Higher girths
# Sketch of proof: $\omega_{r(m,n)}(\mathcal{T}_g^2) = p + 1$



# $\mathcal{T}_g^2$, $g \geq 5$: Higher girths
# Sketch of proof: $\omega_{r(m,n)}(\mathcal{T}_g^2) = p + 1$



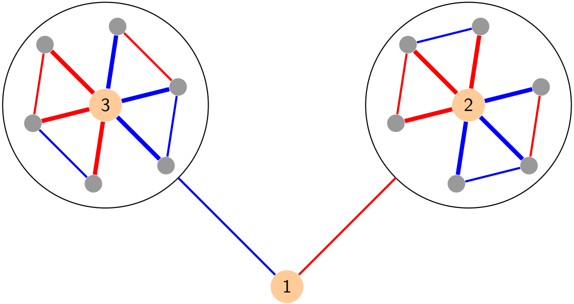
# $\mathcal{T}_g^2$, $g \geq 5$: Higher girths
# Sketch of proof: $\omega_{r(m,n)}(\mathcal{T}_g^2) = p + 1$



# Sketch of proof: $\mathcal{P}_g$

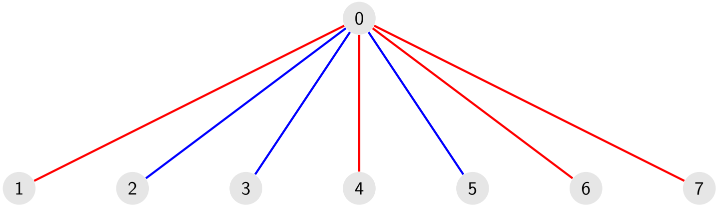
$$3p^2 + p + 1 \leq \omega_{a(m,n)}(\mathcal{P}_3) \leq \omega_{r(m,n)}(\mathcal{P}_3) \leq 42p^2 - 11.$$

# $\mathcal{P}_3$: Planar graphs
# Sketch of proof: The lower bound $(3p^2 + p + 1)$

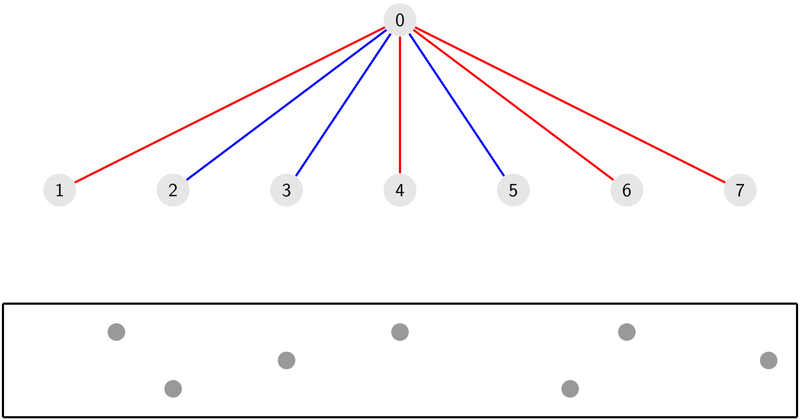



$G$: $(0, 2)$-planar. $p = 2$. $\omega_{a(0,2)}(G) = 3 \times 2^2 + 2 + 1 = 15$

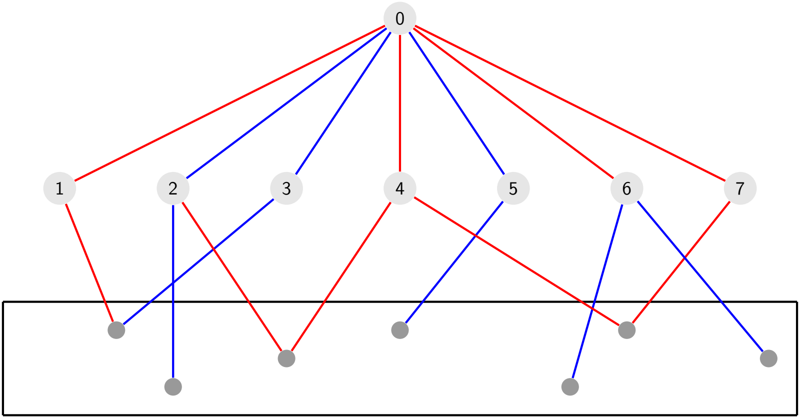
# $\mathcal{P}_3$: Planar graphs
## Sketch of proof: The upper bound $(42p^2 - 13)$



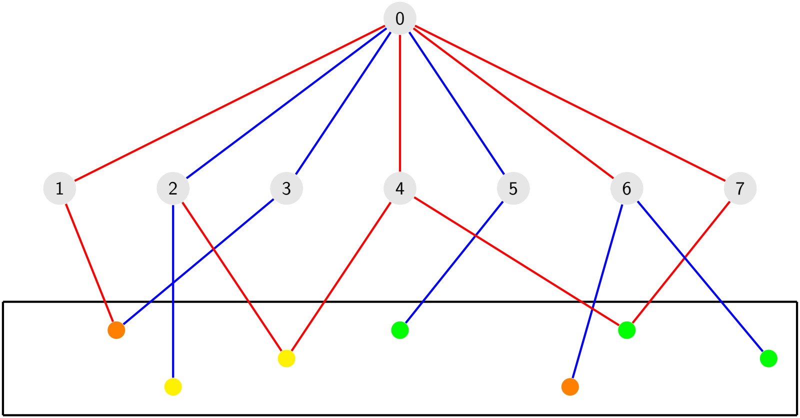
## $\mathcal{P}_3$: Planar graphs
## Sketch of proof: The upper bound $(42p^2 - 13)$



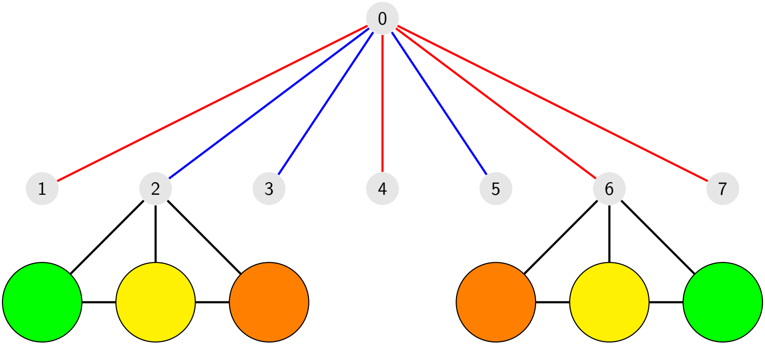
# $\mathcal{P}_3$: Planar graphs
# Sketch of proof: The upper bound $(42p^2 - 13)$



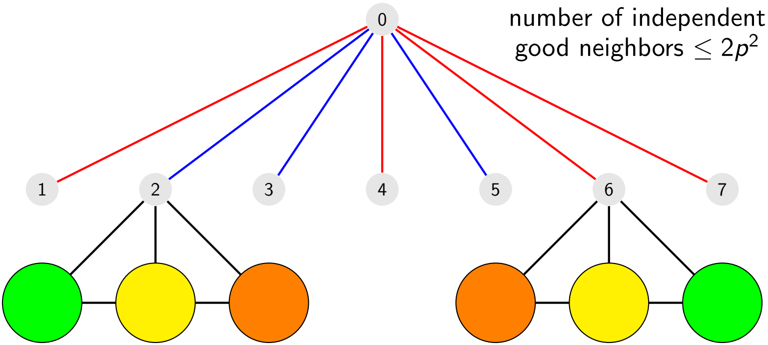
# $\mathcal{P}_3$: Planar graphs
# Sketch of proof: The upper bound $(42p^2 - 13)$



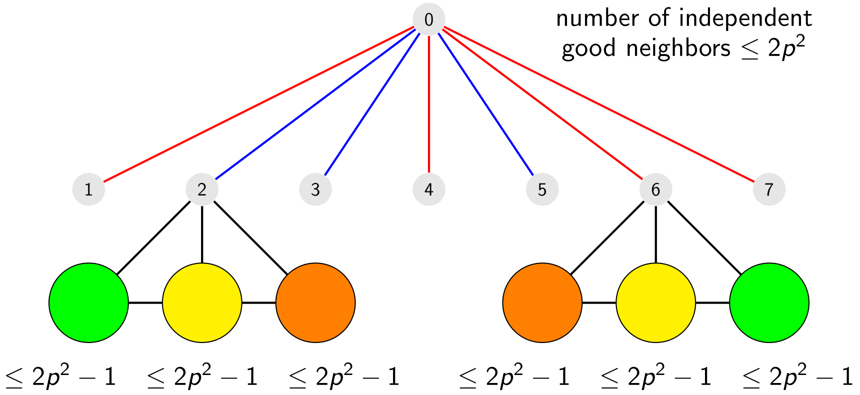
# $\mathcal{P}_3$: Planar graphs
# Sketch of proof: The upper bound $(42p^2 - 13)$



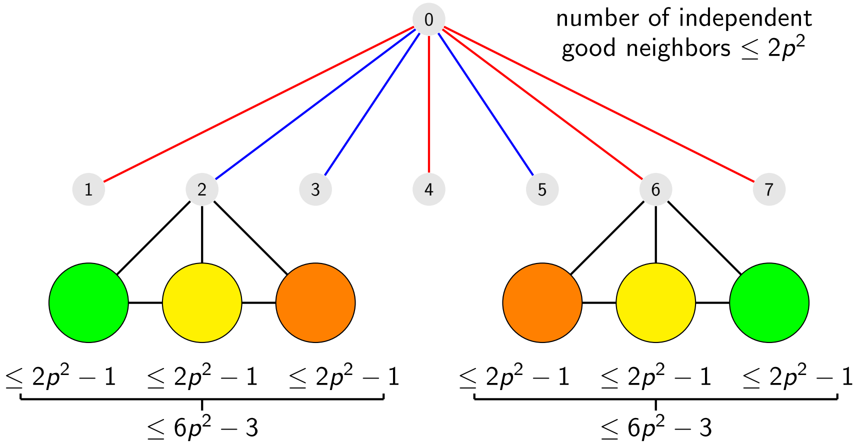
# $\mathcal{P}_3$: Planar graphs
## Sketch of proof: The upper bound $(42p^2 - 13)$



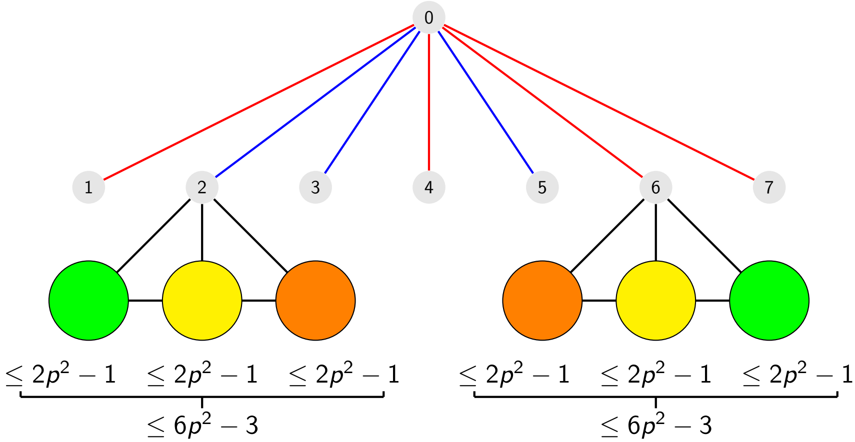
## $\mathcal{P}_3$: Planar graphs
## Sketch of proof: The upper bound $(42p^2 - 13)$



# $\mathcal{P}_3$: Planar graphs
# Sketch of proof: The upper bound $(42p^2 - 13)$

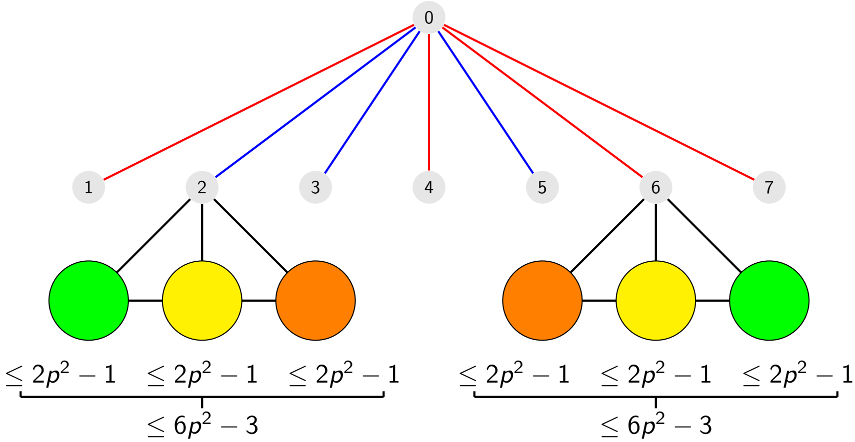


# $\mathcal{P}_3$: Planar graphs
# Sketch of proof: The upper bound $(42p^2 - 13)$



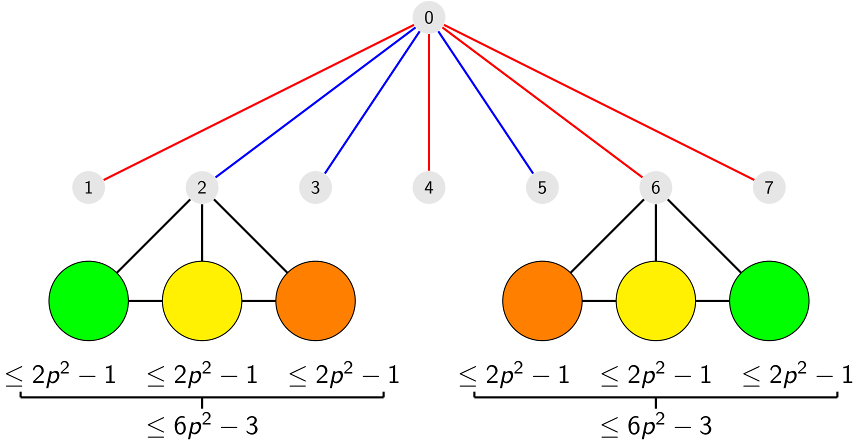

$|R| \leq$

# $\mathcal{P}_3$: Planar graphs
# Sketch of proof: The upper bound $(42p^2 - 13)$



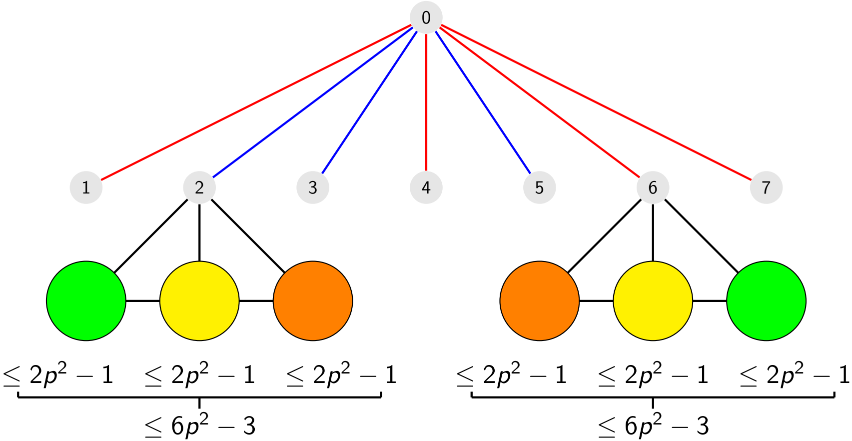

$|R| \leq 1 + 7$

# $\mathcal{P}_3$: Planar graphs
# Sketch of proof: The upper bound $(42p^2 - 13)$




$|R| \leq 1 + 7 + 7 \times (6p^2 - 3)$

# $\mathcal{P}_3$: Planar graphs
# Sketch of proof: The upper bound ($42p^2 - 13$)




$|R| \leq 1 + 7 + 7 \times (6p^2 - 3) = 42p^2 - 13.$

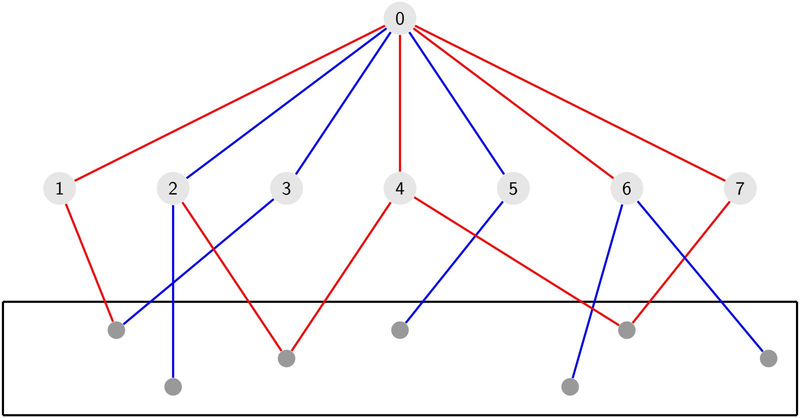
# Sketch of proof: $\mathcal{P}_4$

$$p^2 + 2 = \omega_{a(m,n)}(\mathcal{P}_4) \leq \omega_{r(m,n)}(\mathcal{P}_4) \leq 14p^2 + 1$$

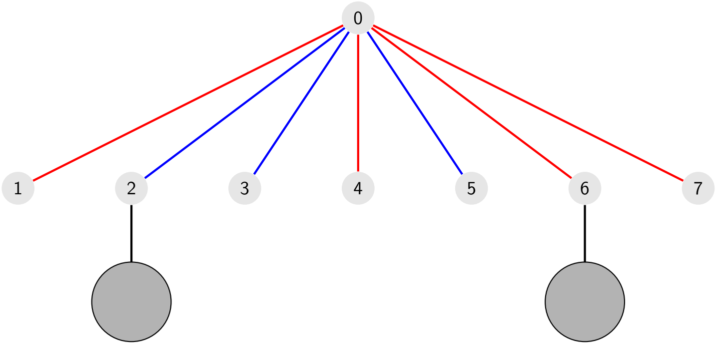
# $\mathcal{P}_4$: Triangle-free
# Sketch of proof: The lower bound $(p^2 + 2)$

Use the same graph as for triangle-free partial 2-tree!

# $\mathcal{P}_4$: Triangle-free
# Sketch of proof: The upper bound $(14p^2 + 1)$



## $\mathcal{P}_4$: Triangle-free
## Sketch of proof: The upper bound $(14p^2 + 1)$




$|R| \leq 1 + 7 + 7 \times (2p^2 - 1) = 14p^2 + 1.$

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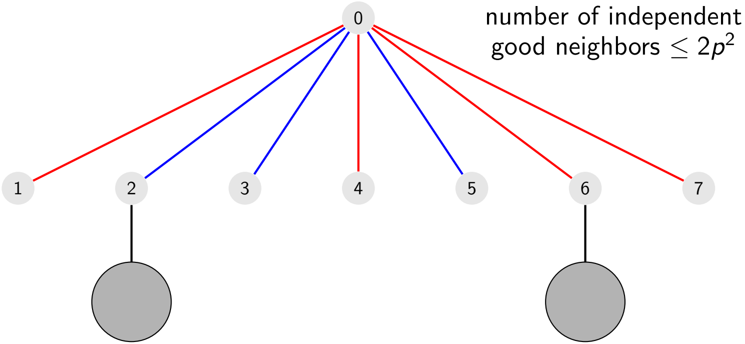


$$|R| \leq 1 + 7 + 7 \times (2p^2 - 1) = 14p^2 + 1.$$

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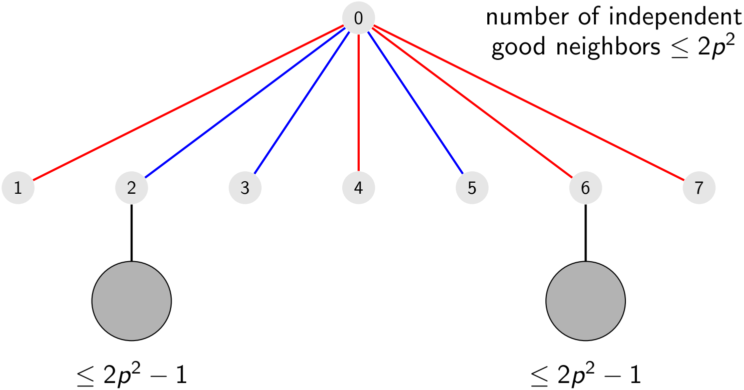

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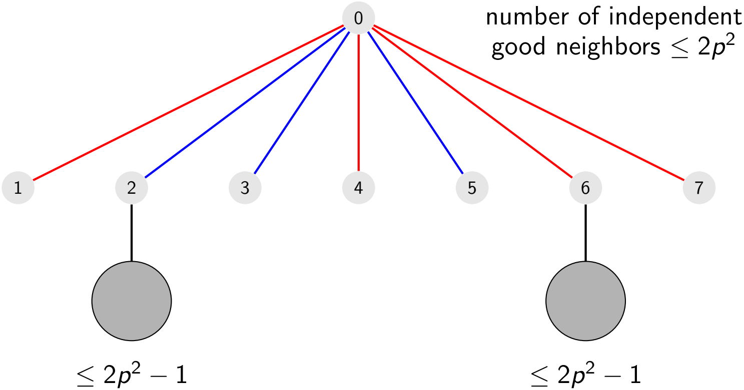


$|R| \leq$

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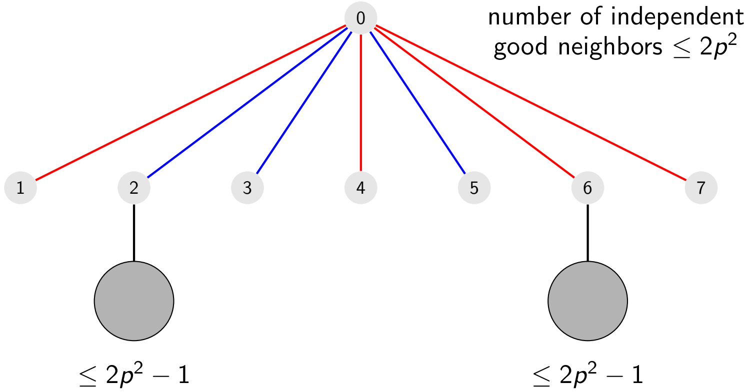


$|R| \leq 1 + 7$

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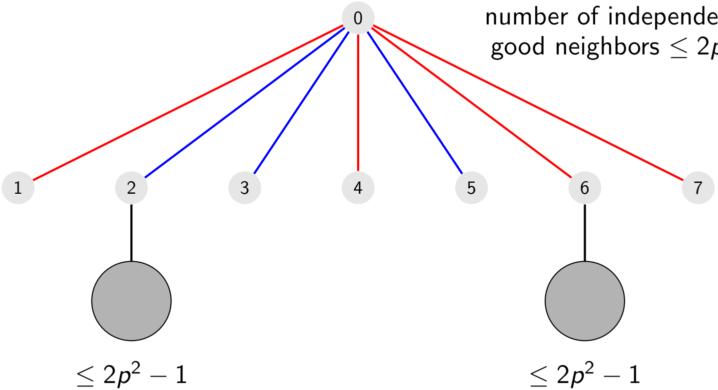


number of independent good neighbors $\leq 2p^2$

$|R| \leq 1 + 7 + 7 \times (2p^2 - 1)$

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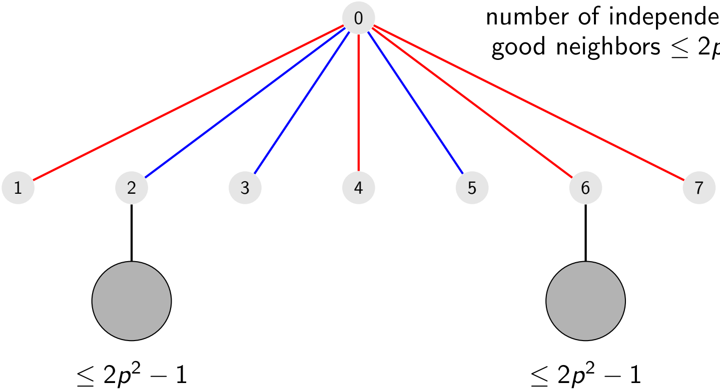


number of independent good neighbors $\leq 2p^2$

$|R| \leq 1 + 7 + 7 \times (2p^2 - 1) = 14p^2 + 1.$

# Sketch of proof: $\mathcal{P}_g$, $g \geq 5$ (higher girths)

$$\omega_{a(m,n)}(\mathcal{P}_g) = \omega_{r(m,n)}(\mathcal{P}_g) = p + 1 \text{ for } p \geq 5$$

# $\mathcal{P}_g$, $g \geq 5$: Higher girths
# Sketch of proof: $\omega_{r(m,n)}(\mathcal{P}_g) = p + 1$

Same logic as for partial 2-trees of higher girths!