On full-separating sets in graphs

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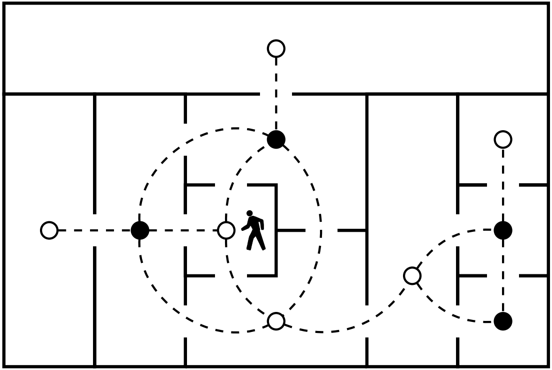
Identification problems in graphs – Locate the intruder



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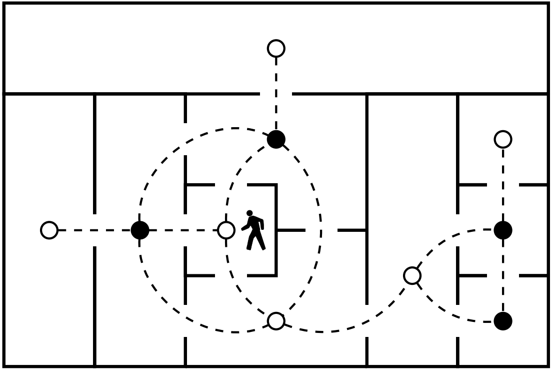

Dominating set: $N[v] \cap C \neq \emptyset$

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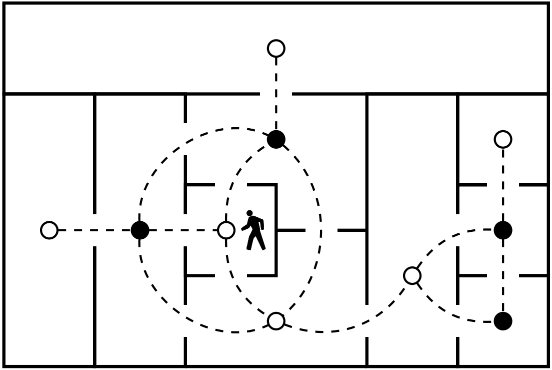

Dominating set: $N[v] \cap C \neq \emptyset$

Total-dominating set: $N(v) \cap C \neq \emptyset$

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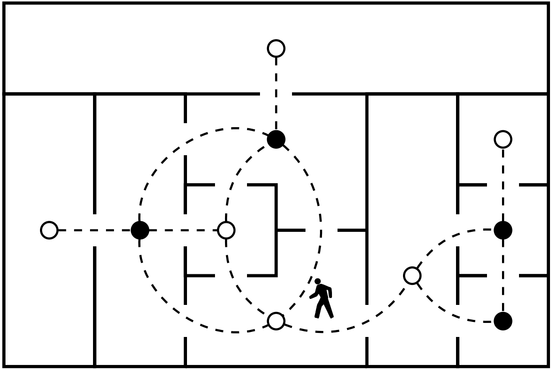

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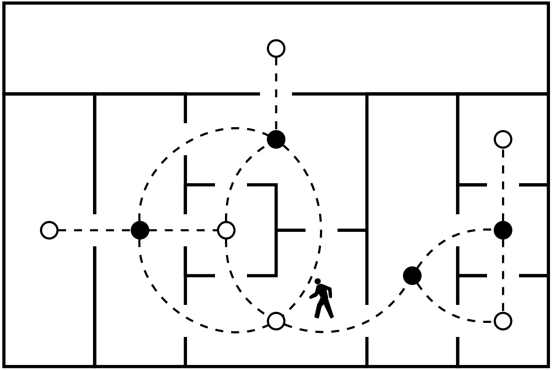

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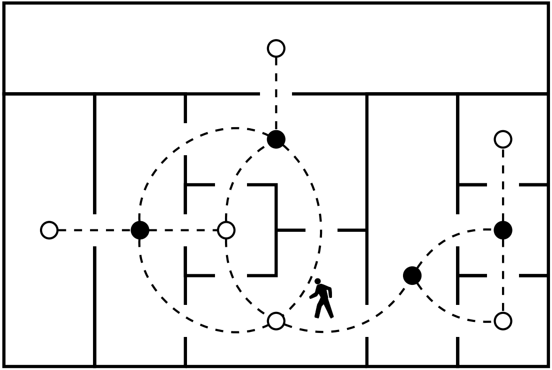

Dominating set: $N[v] \cap C \neq \emptyset$

Total-dominating set: $N(v) \cap C \neq \emptyset$

$N(u) \cap C \neq N(v) \cap C \iff \big(N(u) \triangle N(v)\big) \cap C \neq \emptyset$ for all $u, v \in V \setminus C$

In summary...

Locating set: C such that $\big(N(u) \triangle N(v)\big) \cap C \neq \emptyset$ for all $u, v \in V \setminus C$

No faults: Detectors can distinguish between its vertex and its neighbor

Closed-separating set: C such that $\big(N[u] \triangle N[v]\big) \cap C \neq \emptyset$ for all $u, v \in V$

Fault type 1: Detectors can distinguish between its vertex and its neighbor

Open-separating set: C such that $\big(N(u) \triangle N(v)\big) \cap C \neq \emptyset$ for all $u, v \in V$

Fault type 2: Detector is completely disabled / destroyed

X	LD	LTD	ID	ITD	OD	OTD	**FD**	**FTD**
D/TD	$N[v]$	$N(v)$	$N[v]$	$N(v)$	$N[v]$	$N(v)$	$\mathbf{N[v]}$	$\mathbf{N(v)}$
adj	$\triangle(u,v)$	$\triangle(u,v)$	$\triangle[u,v]$	$\triangle[u,v]$	$\triangle(u,v)$	$\triangle(u,v)$	$\triangle[\mathbf{u},\mathbf{v}]$	$\triangle[\mathbf{u},\mathbf{v}]$
n-adj	$\triangle[u,v]$	$\triangle[u,v]$					$\triangle(\mathbf{u},\mathbf{v})$	$\triangle(\mathbf{u},\mathbf{v})$

$\triangle(u,v) = N(u) \triangle N(v)$; $\triangle[u,v] = N[u] \triangle N[v]$

In summary...

Locating set: C such that $\big(N(u) \triangle N(v)\big) \cap C \neq \emptyset$ for all $u, v \in V \setminus C$

No faults: Detectors can distinguish between its vertex and its neighbor

Equivalently... **Locating set**: C such that

- $\big(N(u) \triangle N(v)\big) \cap C \neq \emptyset$, where u, v adjacent
- $\big(N[u] \triangle N[v]\big) \cap C \neq \emptyset$, where u, v non-adjacent

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Closed-separating set: C such that $\big(N[u] \triangle N[v]\big) \cap C \neq \emptyset$ for all $u, v \in V$

Fault type 1: Detectors can distinguish between its vertex and its neighbor

Open-separating set: C such that $\big(N(u) \triangle N(v)\big) \cap C \neq \emptyset$ for all $u, v \in V$

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<table>
<tr><th>X</th><th>LD</th><th>LTD</th><th>ID</th><th>ITD</th><th>OD</th><th>OTD</th><th>FD</th><th>FTD</th></tr>
<tr><td>D/TD</td><td>$N[v]$</td><td>$N(v)$</td><td>$N[v]$</td><td>$N(v)$</td><td>$N[v]$</td><td>$N(v)$</td><td>$\mathbf{N}[\mathbf{v}]$</td><td>$\mathbf{N}(\mathbf{v})$</td></tr>
<tr><td>adj</td><td>$\triangle(u, v)$</td><td>$\triangle(u, v)$</td><td rowspan="2">$\triangle[u, v]$</td><td rowspan="2">$\triangle[u, v]$</td><td rowspan="2">$\triangle(u, v)$</td><td rowspan="2">$\triangle(u, v)$</td><td>$\triangle[\mathbf{u}, \mathbf{v}]$</td><td>$\triangle[\mathbf{u}, \mathbf{v}]$</td></tr>
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</table>

$\triangle(u, v) = N(u) \triangle N(v)$; $\triangle[u, v] = N[u] \triangle N[v]$

Definitions

Full-separating set: C such that

- $N[u] \cap C \neq N[v] \cap C$, for all $u, v \in V$
- $N(u) \cap C \neq N(v) \cap C$, for all $u, v \in V$

Takes care of Fault type 1 and Fault type 2

Full-separating dominating code (FD-code):
A set with full-separating property + dominating property

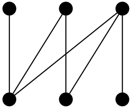
Full-separating total-dominating code (FTD-code):
A set with full-separating property + total-dominating property

Existence & Examples

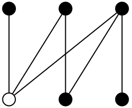
Existence: twin-free = open-twin-free + closed-twin-free

$\mathrm{X} \in \textsc{Codes} = \{\mathrm{LD}, \mathrm{LTD}, \mathrm{ID}, \mathrm{ITD}, \mathrm{OD}, \mathrm{OTD}, \mathrm{FD}, \mathrm{FTD}\}$

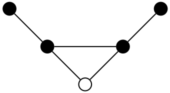
$\gamma^{\mathrm{X}}(G) = \min\{|C| : C \text{ is an X-code of } G\}$



(a) $\gamma^{\mathrm{FTD}}(H_3) = 6$



(b) $\gamma^{\mathrm{FD}}(H_3) = 5$



(c) $\gamma^{\mathrm{FTD}}(B) = \gamma^{\mathrm{FD}}(B) = 4$

Code comparisons

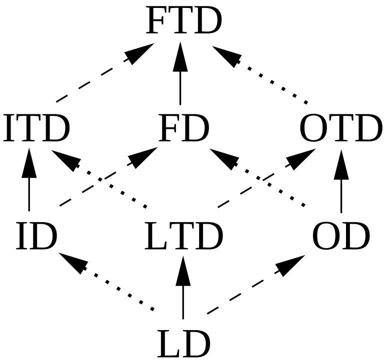


Figure: $X' \longrightarrow X$ stands for $\gamma^{\mathrm{X}'}(G) \leq \gamma^{\mathrm{X}}(G)$

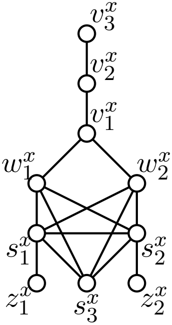
Interesting fact: $\gamma^{\mathrm{FTD}}(G) - 1 \leq \gamma^{\mathrm{FD}}(G) \leq \gamma^{\mathrm{FTD}}(G)$

NP-hardness results related to FD- and FTD-codes

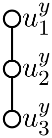
FD
Input: (G, k): A graph G and a positive integer k.
Question: Does there exist an FD-code C of G such that $|C| \leq k$?

FTD
Input: (G, k): A graph G and a positive integer k.
Question: Does there exist an FTD-code C of G such that $|C| \leq k$?

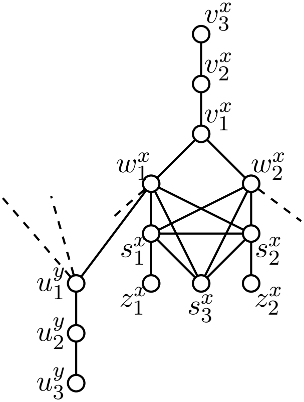
FD = FTD − 1
Input: A graph G and an integer k.
Question: Is $\gamma^{\text{FTD}}(G) = k$ and $\gamma^{\text{FD}}(G) = k - 1$?



(a) Variable gadget G^x.



(b) Clause gadget P^y.

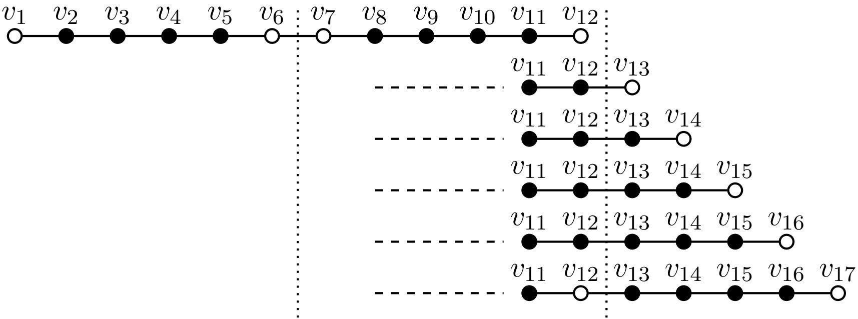


(c) Instance G^ψ.

Theorem

Let G be either a path P_n for $n \geq 4$ or a cycle C_n for $n \geq 5$. Moreover, let $n = 6q + r$ for non-negative integers q and $r \in [0,5]$. Then

$$\gamma^{\mathrm{FD}}(G) = \gamma^{\mathrm{FTD}}(G) = \begin{cases} 4q + r, & \text{if } r \in [0,4]; \\ 4q + 4, & \text{if } r = 5. \end{cases}$$



Thank you!