Identification Problems in Graphs

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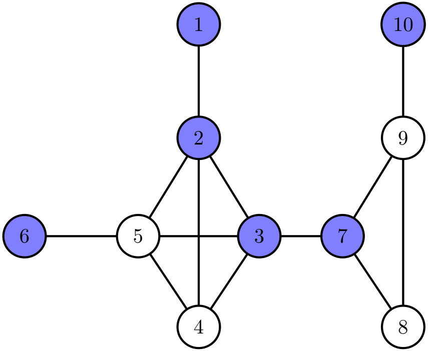
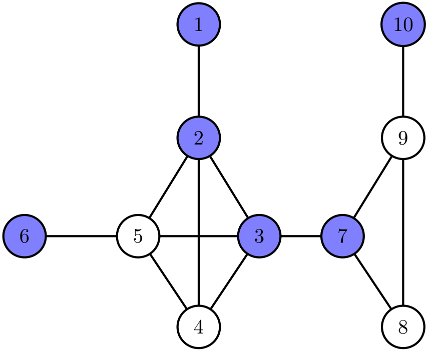
1. Introduction to Identification problems
   - Motivation
   - Definitions
   - Properties

2. LD and LTD codes
   - Bounds and existing research
   - Recent developments

Identification problems in graphs

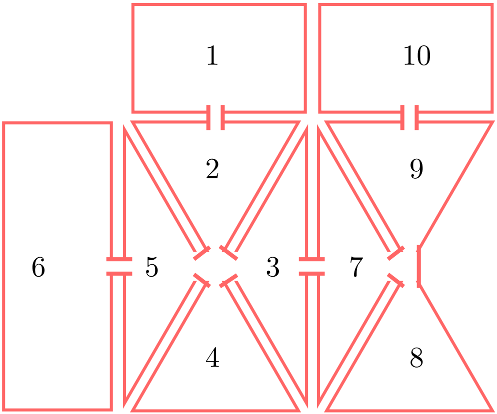


Identification problems in graphs



- $4 \longleftrightarrow \{2, 3\}$
- $5 \longleftrightarrow \{2, 3, 6\}$
- $8 \longleftrightarrow \{7\}$
- $9 \longleftrightarrow \{7, 10\}$

Locating-dominating (LD) code – A practical example



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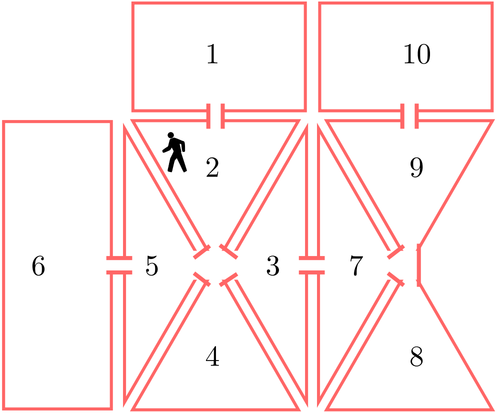

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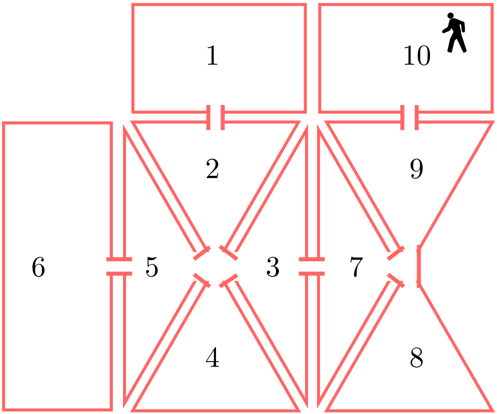

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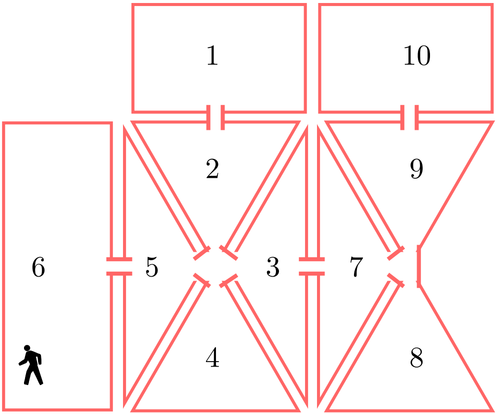

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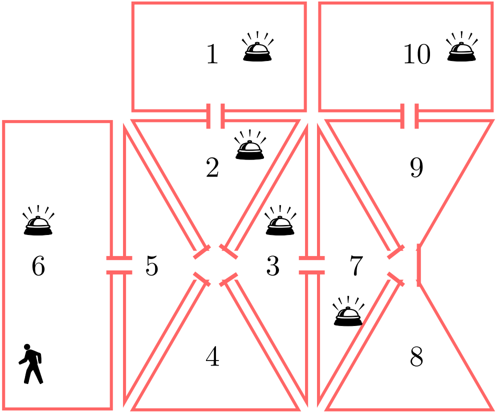

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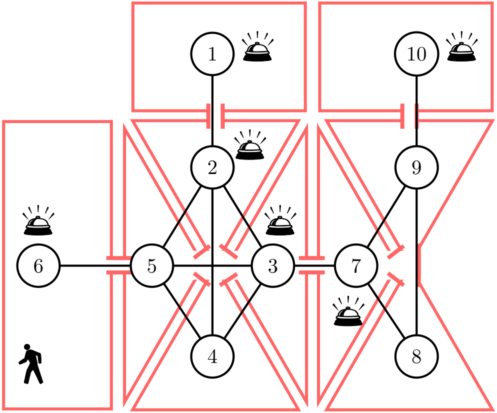

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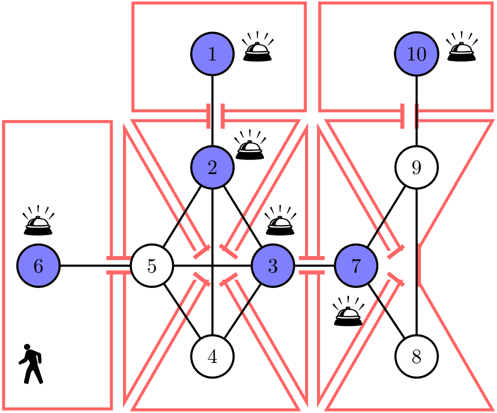

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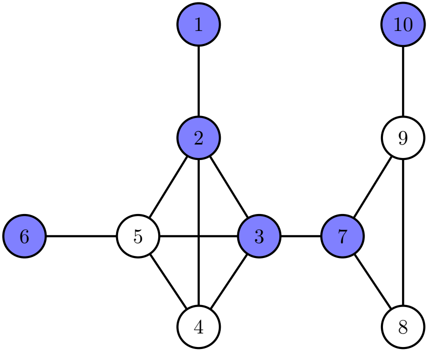

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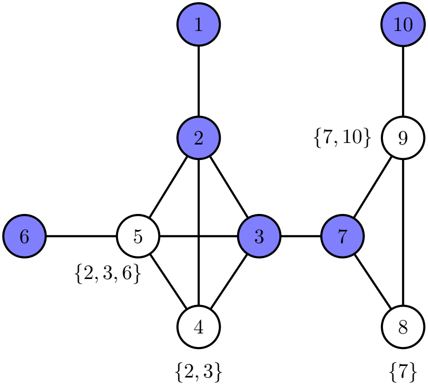

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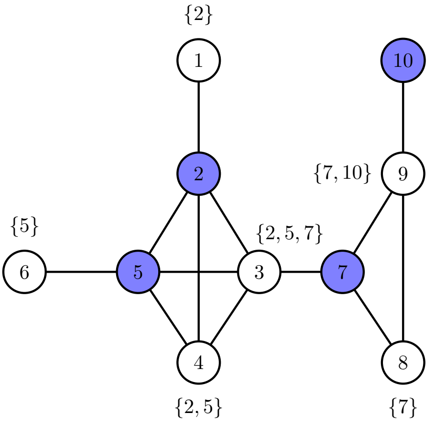

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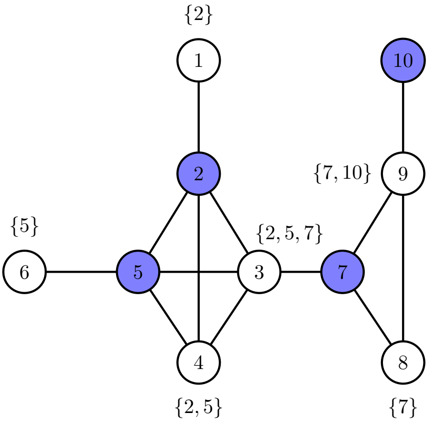

$S = \{\text{blue vertices}\}$

- $1 \longleftrightarrow \{2\} = N(1) \cap S$
- $3 \longleftrightarrow \{2, 5, 7\} = N(3) \cap S$
- $4 \longleftrightarrow \{2, 5\} = N(4) \cap S$
- $6 \longleftrightarrow \{5\} = N(6) \cap S$
- $8 \longleftrightarrow \{7\} = N(8) \cap S$
- $9 \longleftrightarrow \{7, 10\} = N(9) \cap S$

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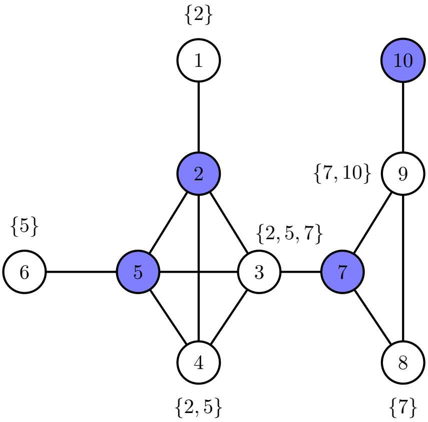

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S has two properties:

1. Domination: "Well spread-out"
2. Separation (*location*):

$N(u) \cap S \neq N(v) \cap S, \forall u, v \notin S$

Definitions (Neighbourhoods)

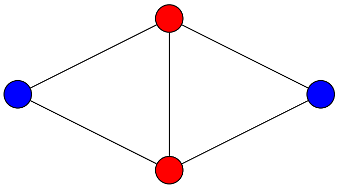
Let $G = (V, E)$ be a graph...

Neighbourhoods

open: $N(v) = \{u \in V : uv \in E\}$. || *closed*: $N[v] = N(v) \cup \{v\}$.

Twins

$$u, v \in V \text{ are called } \begin{Bmatrix} \text{open twins} \\ \text{closed twins} \end{Bmatrix} \iff \begin{Bmatrix} N(u) = N(v) \\ N[u] = N[v] \end{Bmatrix}.$$



Diamond

Definitions (Domination)

Let $G = (V, E)$ be a graph...

Dominating set [property: *closed domination*]

$S \subset V$ such that $N[v] \cap S \neq \emptyset$ for all $v \in V$.

A dominating set always exists.

Open/total dominating set [property: *open domination*]

$S \subset V$ such that $N(v) \cap S \neq \emptyset$ for all $v \in V$.

An open dominating set exists $\iff$ G is isolate-free.

Definitions (Separation)

Let $G = (V, E)$ be a graph...

Locating set [property: *location*]

$S \subset V$ such that $N(u) \cap S \neq N(v) \cap S$ for all $u, v \in V \setminus S$.

Open separating set [property: *open separation*]

$S \subset V$ such that $N(u) \cap S \neq N(v) \cap S$ for all distinct $u, v \in V$.

Open separating set exists $\iff$ G is open-twin free.

Closed separating set [property: *closed separation*]

$S \subset V$ such that $N[u] \cap S \neq N[v] \cap S$ for all distinct $u, v \in V$.

Closed separating set exists $\iff$ G is closed-twin free.

Definitions (Codes)

Let $G = (V, E)$ be a graph...

Locating-dominating (LD) code [3] X=LD

$S \subset V$ such that S has the following properties.

- domination : closed domination
- separation : location

LD code always exists.

[3] Slater, 1988

Definitions (Codes)

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LD code always exists.

X code

$S \subset V$ such that S has the following properties.

- domination (X) : closed / open domination
- separation (X) : location / closed / open separation

[3] Slater, 1988

Definitions (Codes)

Let $G = (V, E)$ be a graph...

Locating total-dominating (LTD) code [4] X=LTD

$S \subset V$ such that S has the following properties.

- domination : open domination
- separation : location

LTD code exists $\iff$ G is isolate-free.

[4] Haynes, Henning & Howard, 2006.

X codes studied in literature.

Code name	**X**	**dom(X)**	**sep(X)**	$\gamma^{\mathbf{X}}(\mathbf{G})$
Identifying	ID	CD	CS	$\gamma^{\text{ID}}(G)$
Open-Separating Dominating	OSD	CD	OS	$\gamma^{\text{OSD}}(G)$
Locating- Dominating	LD	CD	L	$\gamma^{\text{LD}}(G)$
Differentiating Total-Dominating	DTD	OD	CS	$\gamma^{\text{DTD}}(G)$
Open-Locating Dominating	OLD	OD	OS	$\gamma^{\text{OLD}}(G)$
Locating Total-Dominating	LTD	OD	L	$\gamma^{\text{LTD}}(G)$

CD : closed domination
OD : open domination
CS : closed separation
OS : open separation
L : location

Definitions

Let $G = (V, E)$ be a graph...

X-number

$\gamma^{\mathrm{X}}(G) = \min\{|S| : S \text{ is an X-code of } G\}$.

Domination number

$\gamma(G) = \min\{|S| : S \text{ is a dominating set of } G\}$.

Total-domination number

$\gamma_t(G) = \min\{|S| : S \text{ is a total-dominating of } G\}$.

Basic properties of X codes

Let $G = (V, E)$ be a graph...

A comparison

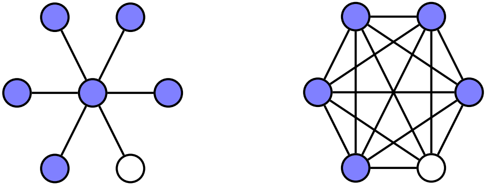
$$\gamma^{\mathrm{X}}(G) \geq \begin{cases} \gamma(G), & \text{if domination (X)=CD} \\ \gamma_t(G), & \text{if domination (X)=OD} \end{cases}$$

$G = G_1 \sqcup G_2 \sqcup \ldots \sqcup G_k$

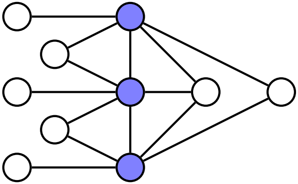
$$\gamma^{\mathrm{X}}(G) = \gamma^{\mathrm{X}}(G_1) + \gamma^{\mathrm{X}}(G) \ldots + \gamma^{\mathrm{X}}(G_k) \text{ except when } \mathrm{X} = \mathrm{OSD}.$$

$\implies$ enough to consider connected G whenever $\mathrm{X} \neq \mathrm{OSD}$.

Codes based on location: LD codes...

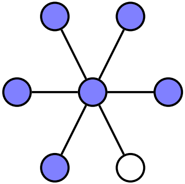


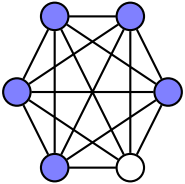
$\gamma^{\mathrm{LD}}(G) = n - 1$, $\gamma(G) = 1$.



$n = 10$, $\gamma^{\mathrm{LD}}(G) = 3 = \lfloor \log_2 10 \rfloor$.

Codes based on location: LD codes...





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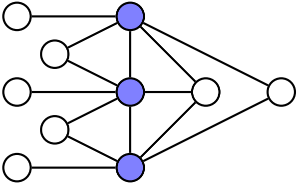

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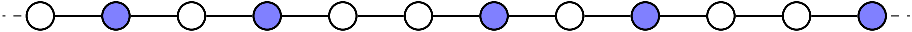
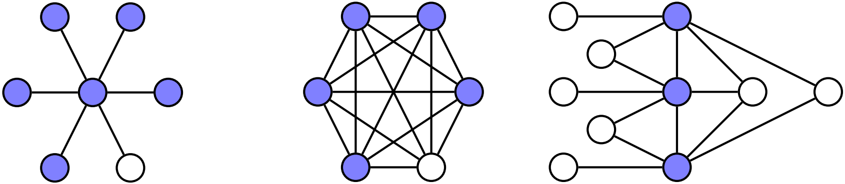


$$\gamma^{LD}(P_n) = \lceil \tfrac{2}{5} n \rceil$$

Codes based on location: LD codes...



$\gamma^{\mathrm{LD}}(G) = n - 1$, $\gamma(G) = 1$. $n = 10$, $\gamma^{\mathrm{LD}}(G) = 3 = \lfloor \log_2 10 \rfloor$.

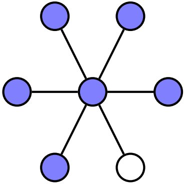
Let $G = (V, E)$ be a connected graph and of order ≥ 3.

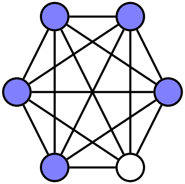
LD code: $S \subset V$ is a **dominating set** and a **locating set**.

- $\gamma(G) \leq \gamma^{\mathrm{LD}}(G)$
- $\lfloor \log_2 n \rfloor \leq \gamma^{\mathrm{LD}}(G) \leq n - 1$ [3]

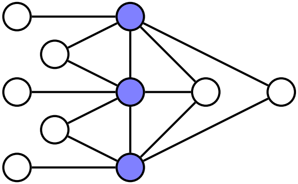
[3] Slater, 1988.

Codes based on location: LTD codes...





$\gamma^{\mathrm{LTD}}(G) = n - 1$, $\gamma(G) = 1$.



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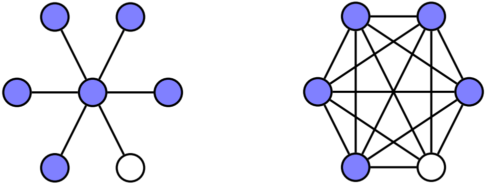
Codes based on location: LTD codes...



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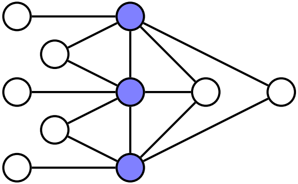

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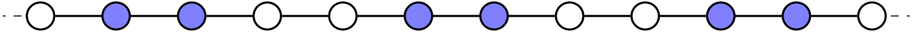
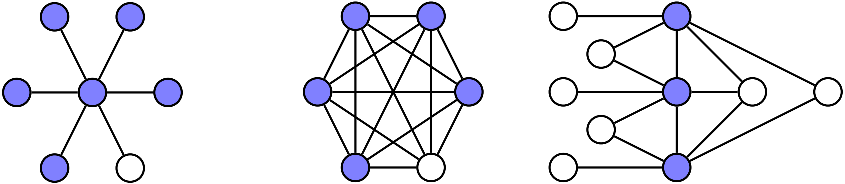


$$\gamma^{\mathrm{LTD}}(P_n) = \lfloor \tfrac{n}{2} \rfloor + \lceil \tfrac{n}{4} \rceil - \lfloor \tfrac{n}{4} \rfloor$$



$\gamma^{\text{LTD}}(G) = n - 1$, $\gamma(G) = 1$. $n = 10$, $\gamma^{\text{LTD}}(G) = 3 = \lfloor \log_2 10 \rfloor$.

Let $G = (V, E)$ be a connected graph and of order ≥ 3.

LD code: $S \subset V$ is a **total-dominating set** and a **locating set**.

- $\gamma_t(G) \leq \gamma^{\text{LTD}}(G)$
- $\lfloor \log_2 n \rfloor \leq \gamma^{\text{LTD}}(G) \leq n - 1$ [5]

[5] Henning & Rad, 2012.

Bounds on $\gamma^{\mathrm{LD}}(G)$ and $\gamma^{\mathrm{LTD}}(G)$

LD code: $S \subset V$ is a

- dominating set; and
- locating set

$$\gamma(G) \leq \gamma^{\mathrm{LD}}(G)$$
$$\gamma(G) \leq \frac{n}{2} \quad ([6])$$
$$\gamma(G) \leq \gamma^{\mathrm{LD}}(G) \leq \frac{n}{2}$$

(conjecture [7]: for G twin-free)

[6] Ore, 1962.
[7] Garijo, González & Márques, 2014.

LTD code: $S \subset V$ is a

- total-dominating set; and
- locating set

$$\gamma_t(G) \leq \gamma^{\mathrm{LTD}}(G)$$
$$\gamma_t(G) \leq \frac{2n}{3} \quad ([8])$$
$$\gamma_t(G) \leq \gamma^{\mathrm{LTD}}(G) \leq \frac{2n}{3}$$

(conjecture [9]: for G twin-free)

[8] Cockayne, Dawes & Hedetniemi, 1980.
[9] Foucaud & Henning, 2016.

Conj: $\gamma^{\mathrm{LD}}(G) \leq \frac{n}{2}$, G isolate- & twin-free, is true for:

• with no 4-cycles (as subgraphs) • with independence number $\geq \frac{n}{2}$ • bipartite graph • with vertex cover number $\leq \frac{n}{2}$	Garijo, González & Márquez, 2014
• line graph • cubic graph (Conj: even with twins)	Foucaud & Henning, 2016
• subcubic graph (with deg 1 twin-free) $\not\cong K_3, K_4, K_{2,3}, K_{3,3}$ • Corollary: cubic with twins	C., Lehtila & Hakanen, 2024+ [writing ongoing]
• split graph • co-bipartite	Foucaud, Henning, Löwenstein & Sasse, 2016
• block graph	C., Foucaud, Parreau & Wagler, 2023

Best general bound [Foucaud, Henning, Löwenstein & Sasse 2016]

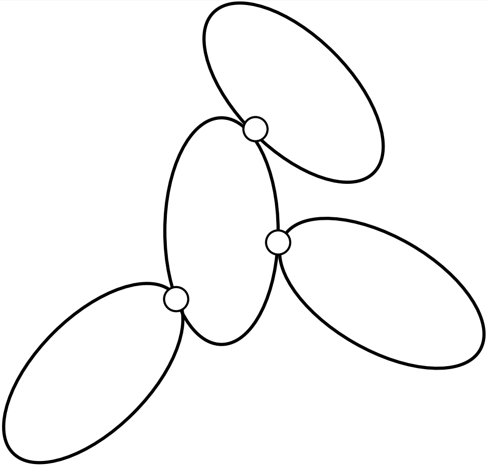
For a twin-free and isolate-free graph, $\gamma^{\mathrm{LD}}(G) \leq \frac{2n}{3}$.

Block graph

A graph whose every 2-connected component is a complete subgraph.

Block graph (altertnative definition)

A diamond-free chordal graph.



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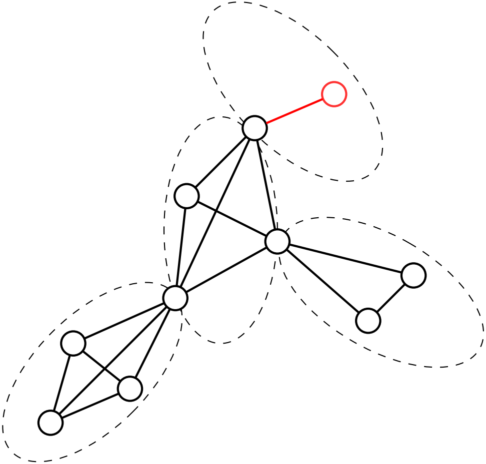

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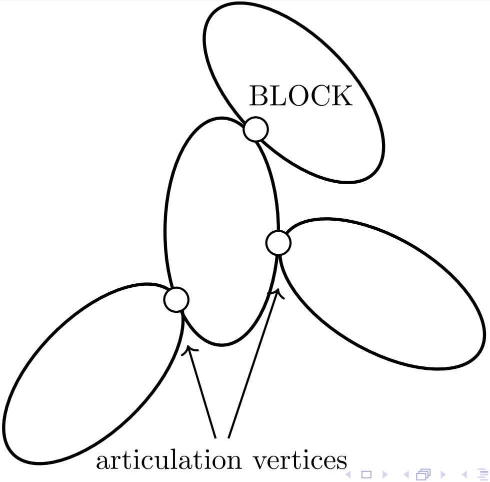

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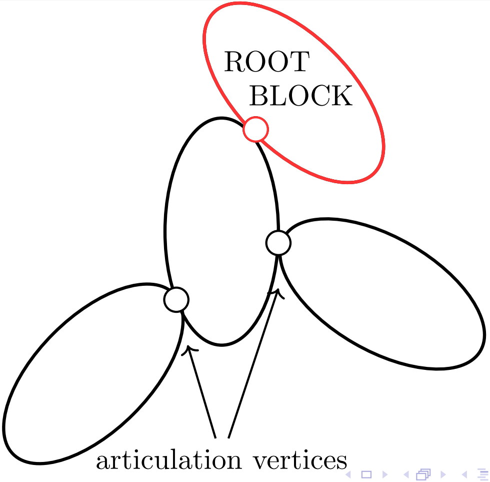

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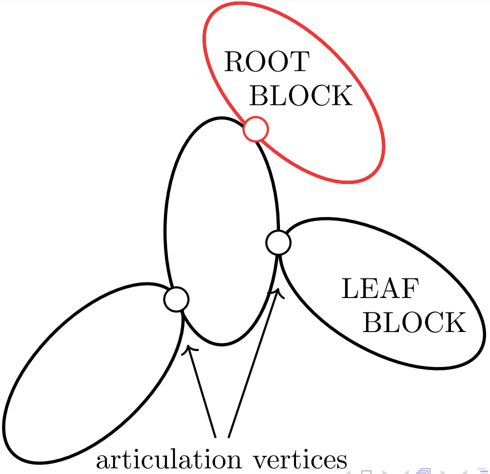

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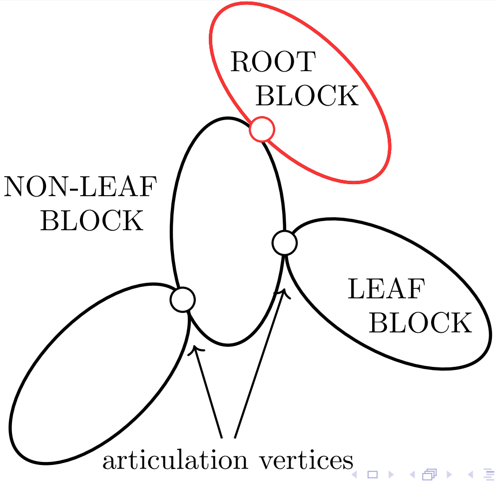

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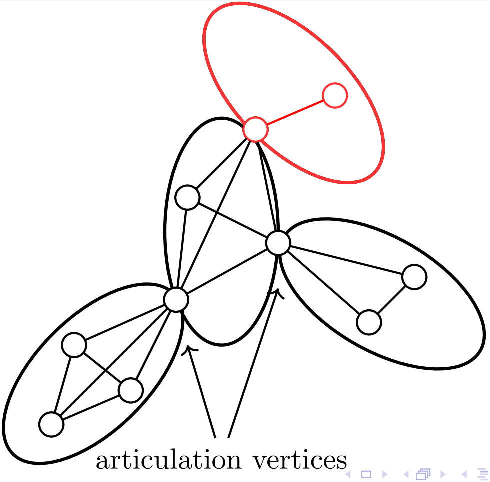

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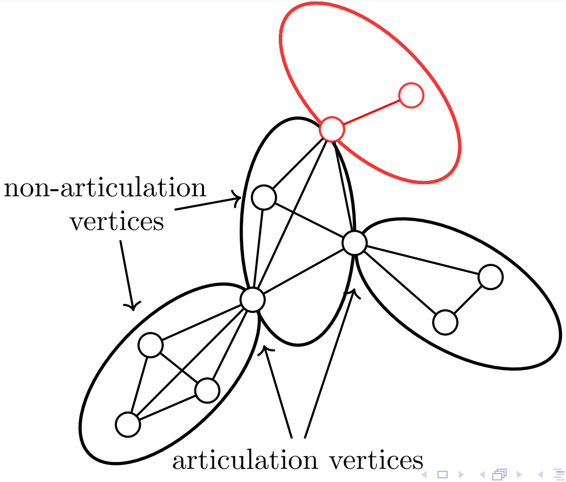

Theorem (C., Foucaud, Parreau & Wagler, 2023)

For G a twin-free and isolate-free block graph on n vertices,

$$\gamma^{LD}(G) \leq \frac{n}{2}.$$

Proof sketch.

□

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Proof sketch.

- Partition V into two parts A and B.

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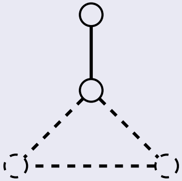

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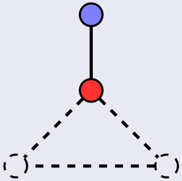

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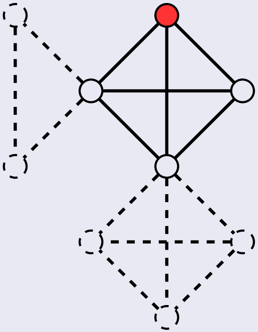
- Partition V into two parts A and B.



Theorem (C., Foucaud, Parreau & Wagler, 2023)

For G a twin-free and isolate-free block graph on n vertices,

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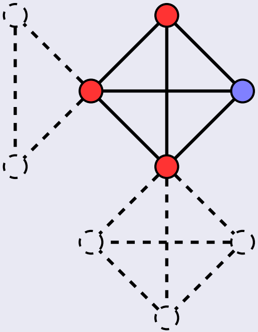

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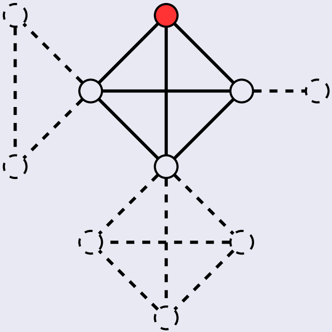

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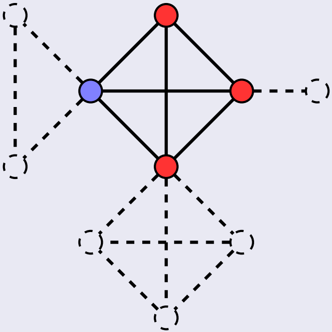

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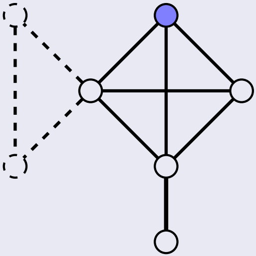

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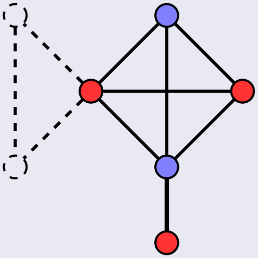

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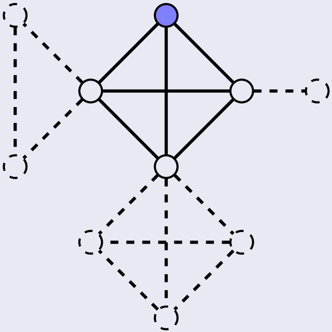

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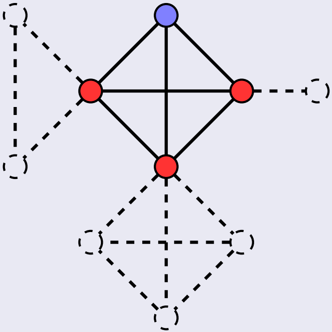

Theorem (C., Foucaud, Parreau & Wagler, 2023)

For G a twin-free and isolate-free block graph on n vertices,

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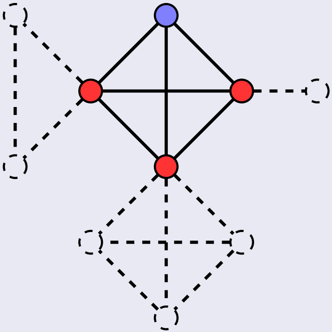
Proof sketch.

- Partition V into two parts A and B.
- Both A and B are LD codes of G.



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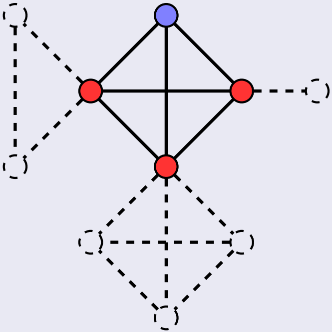
Theorem (C., Foucaud, Parreau & Wagler, 2023)

For G a twin-free and isolate-free block graph on n vertices,

$$\gamma^{LD}(G) \leq \frac{n}{2}.$$

Proof sketch.

- Partition V into two parts A and B.
- Both A and B are LD codes of G.
- Either one of $|A|$ or $|B| \leq \frac{1}{2}n$.



□

Conj: $\gamma^{\text{LTD}}(G) \leq \frac{2n}{3}$, G isolate- & twin-free, is true for:

• with no 4-cycles (as subgraphs) • line graph • with $\gamma^{\text{LD}}(G) \leq \frac{n}{2}$ and $\delta(G) \geq 26$	Foucaud & Henning, 2016
• claw-free cubic graph $\ncong K_4$ (It was shown $\gamma^{\text{LTD}}(G) \leq \frac{n}{2}$. Conj: $\gamma^{\text{LTD}}(G) \leq \frac{n}{2}$ for G cubic, $n \geq 8$)	Henning & Löwenstein, 2012
• split graph ($\gamma^{\text{LTD}}(G) < \frac{2n}{3}$) • co-bipartite graph ($\gamma^{\text{LTD}}(G) \leq \frac{n}{2}$) • block graph, $n \geq 4$ • subcubic graph $\ncong K_1, K_2, K_4, K_{1,3}$ • outerplanar graph	C., Foucaud, Hakanen, Henning & Wagler, 2024+

Best general bound [Foucaud & Henning, 2016]

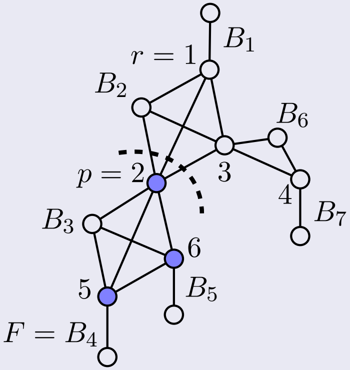
For a twin-free and isolate-free graph, $\gamma^{\text{LTD}}(G) \leq \frac{3n}{4}$.

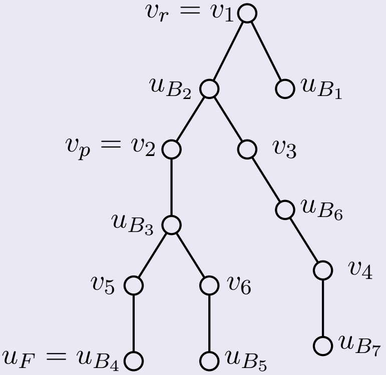
Theorem (C.,Foucaud, Hakanen, Henning & Wagler, 2024+)

For G a twin-free and isolate-free block graph on n vertices,

$$\gamma^{\mathrm{LTD}}(G) \leq \frac{2n}{3}.$$

Proof sketch (by induction on n).





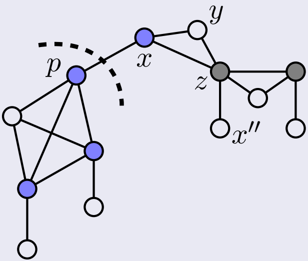
Theorem (C.,Foucaud, Hakanen, Henning & Wagler, 2024+)

For G a twin-free and isolate-free block graph on n vertices,

$$\gamma^{\text{LTD}}(G) \leq \frac{2n}{3}.$$

Proof sketch (by induction on n).

- x and y are (closed) twins in G'.



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For G a twin-free and isolate-free block graph on n vertices,

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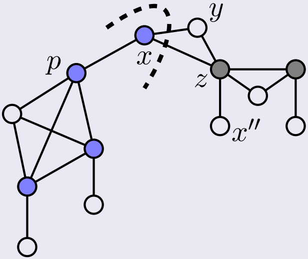
- x and y are (closed) twins in G'.
- y and x'' are (open) twins in G''.



□

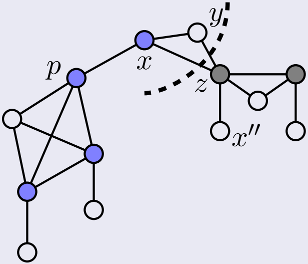
Theorem (C.,Foucaud, Hakanen, Henning & Wagler, 2024+)

For G a twin-free and isolate-free block graph on n vertices,

$$\gamma^{\text{LTD}}(G) \leq \frac{2n}{3}.$$

Proof sketch (by induction on n).

- x and y are (closed) twins in G'.
- y and x'' are (open) twins in G''.
- No further twins.



□

Theorem (C.,Foucaud, Hakanen, Henning & Wagler, 2024+)

For G a connected subcubic graph on n vertices and $\ncong K_1, K_2, K_4, K_{1,3}$,

$$\gamma^{\text{LTD}}(G) \leq \frac{2n}{3}.$$

Proof sketch

- Proof by contradiction: assume G to be subcubic graph of smallest order such that $\gamma^{LTD}(G) > \frac{2}{3}n$.

Theorem (C.,Foucaud, Hakanen, Henning & Wagler, 2024+)

For G a connected subcubic graph on n vertices and $\not\cong K_1, K_2, K_4, K_{1,3}$,

$$\gamma^{\text{LTD}}(G) \leq \frac{2n}{3}.$$

Proof sketch

- Proof by contradiction: assume G to be subcubic graph of smallest order such that $\gamma^{LTD}(G) > \frac{2}{3}n$.
- $\delta(G) \geq 2$.
  - There is no $(1,3,1)$-sequence.
  - There is no $(1,2,2)$-sequence.
  - There is no $(1,2,3,1)$-sequence.
  - There is no $(1,2,3,2,1)$-sequence.
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Theorem (C.,Foucaud, Hakanen, Henning & Wagler, 2024+)

For G a connected subcubic graph on n vertices and $\ncong K_1, K_2, K_4, K_{1,3}$,

$$\gamma^{\text{LTD}}(G) \leq \frac{2n}{3}.$$

Proof sketch

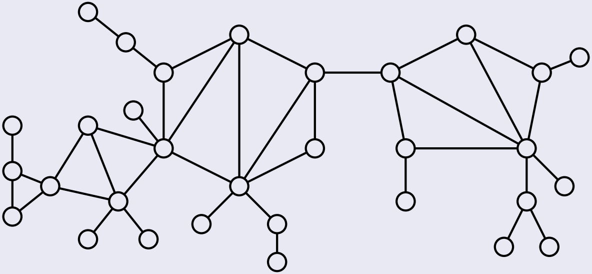
- G is triangle-free.
- G is cubic.
  - There is no $(2,2,2)$-sequence.
  - There is no $(2,3,2)$-sequence.
  - There is no $(2,2,3)$-sequence.
  - There is no $(2,2)$-sequence.
  - There is no $(2,3,3)$-sequence.
- $G' = G - (3,3,3)$-sequence.
- $G' \ncong K_1, K_2, K_4, K_{1,3}$.

Theorem (C.,Foucaud, Hakanen, Henning & Wagler, 2024+)

For G a twin-free and isolate-free outerplanar graph on n vertices,

$$\gamma^{\mathrm{LTD}}(G) \leq \frac{2n}{3}.$$

Proof sketch



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For G a twin-free and isolate-free outerplanar graph on n vertices,

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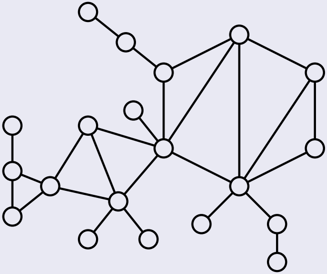
Proof sketch



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For G a twin-free and isolate-free outerplanar graph on n vertices,

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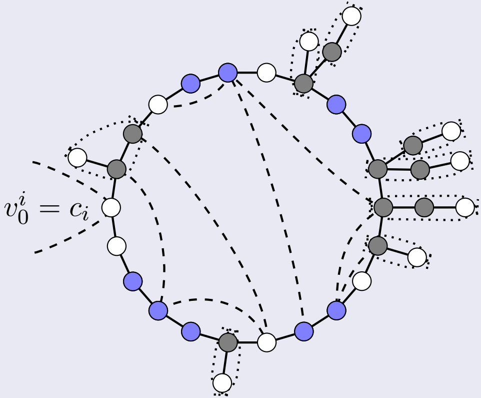
Proof sketch



Possible future research ideas...

- Prove the conjectures in general!
- Improve the existing best bounds approximating the conjectured bounds.
- Characterize cubic/ block / split / etc. graphs whose LD-LTD-numbers attain the conjectured bounds.

Thank you.